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NOTES ON THE HISTORY OF THE SLIDE RULE.*

By FLORIAN CAJORI, Colorado College, Colorado Springs.

Few instruments designed for minimizing mental labor in computation offer a more attractive field for historical study than the slide rule. Its development has reached in many directions and has attracted a great variety of intellect. Not only have writers on arithmetic been drawn to it, but also carpenters, excise officers, practical engineers, chemists, physicists, and mathematicians, including even the great Sir Isaac Newton.

And yet, the history of this instrument has been neglected to such an extent that gross inaccuracies occur in standard publications.

The first point I desire to make relates to the invention of the straight-edge slide rule. One of our American manufacturers of slide rules has an instrument on the market, called the "Gunter slide rule" and claims that it "is the original form of the slide rule." As a matter of fact Gunter never invented a slide rule. What Gunter did do was to publish in 1620, six years after Napier's publication of his logarithms, a work containing a description of Gunter's "line of numbers," which, when mounted upon a scale was called "Gunter's scale." On it distances were taken proportional to the logarithms of numbers; it was logarithms laid off upon straight lines. But Gunter's scale contained no sliding parts and, therefore, was not a slide rule.

Charles Hutton, in his *Mathematical Dictionary* (Art. "Gunter's Line"), and also in his *Mathematical Tables*, ascribes the invention of the slide rule to Edmund Wingate, 1627, but he nowhere substantiates his statement by reference to any of Wingate's works. De Morgan in his article "Slide Rule" in the *Penny Cyclopaedia* (1842), and in later publications, ascribes the invention to William Oughtred, a famous writer of mathematical text-books, 1632, and denies that Wingate ever wrote on the slide rule. It will soon appear that De Morgan was ill-informed on this subject, for he had not seen all of Wingate's works, although his criticism of a passage in Ward's *Lives of the Professors of Gresham College* (1740) is well taken. Ward claims that Wingate introduced the slide rule into France

*Read before the Southwest Section of the American Mathematical Society, in St. Louis, November 30, 1907.

in 1624. What he at that time really did introduce was Gunter's scale, as appears from the examination of his book, published in Paris in 1624. To prove or disprove the claim made for Wingate requires the examination of his numerous writings. To the present writer Wingate's publications are not accessible. An inquiry directed to the Keeper of the Printed Books at the British Museum in London brought the reply that in the work entitled, the *Construction and Use of the Line of Proportion,* London, 1628, the slide rule is explained. Prefixed to the book is a diagram of the "line of proportion," now called slide rule. Wingate says in his preface, "I have invented this tabular scale or line of proportion." Further on he says "the line of proportion is a double scale, broken off in tenne Fractions, upon which Logarithms of numbers are found out." This book was probably reproduced two years later in Wingate's work *Arithmetic made easy, or natural and artificial arithmetic,* London, 1630, a text quoted by Favaro* in his history of the slide rule. A second edition of Wingates publication of 1630 appeared in 1652, wherein improvements in the divisions of the slide rule are described. From these facts it appears that De Morgan was in error, and that the claim made for Edmund Wingate as the inventor of the straight-edge slide rule is well founded, for he published four years earlier than did William Oughtred. It should be added, however, that Oughtred describes also a circular slide rule and that he has a clear title as the inventor of the circular type.

My second point relates to the invention of the "runner." In 1850 a French artillery-officer and mathematician, A. Mannheim, designed a slide rule with a "runner," now generally known as the "Mannheim rule." German writers† have called attention to the fact that Mannheim was not the first inventor of the runner, that a description of it occurs in a French work of 1837, thirteen years earlier. My own reading reveals that the runner was invented much earlier in England and afterwards completely forgotten by the English. The first traces go back to Sir Isaac Newton, but in 1842 even De Morgan who writes at length on the slide rule and its history, makes no reference whatever to the runner. It is not generally known that Sir Isaac Newton referred to the slide rule. In Newton's works is given an extract from a letter of Oldenburg to Leibniz, dated June 24, 1675, which we shall consider more fully later. The "runner" is not mentioned in this extract, but Newton's slide rule could not be used without the employment of some device like that of the "runner." Sixty-eight years later, Newton's scheme slightly modified is explained more fully in Stone's *Mathematical Dictionary,* 2nd Ed., 1743. I am not aware that Newton's and Stone's slide rules were ever actually constructed and used in practice. But thirty-five years after Stone's publication a book was published in London, containing

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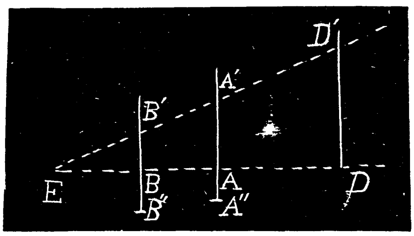
* *Veneto Instituto Atti* (5) 5, 1878-79, p. 495, abbreviated in Favaro's *Lecons de statique graphique,* 2 eme partie, *calcul graphique,* Paris, 1885, translated into the French by P. Terrier.

†*Zeitschrift f. Math. and Phys.,* Vol. 48, 1903, p. 134.

an instrument by John Robertson, which employed the runner and which was constructed in Cornhill by Messrs. Nairne and Blunt, and put upon the market. There are no indications that Robertson's rule ever became popular. Later the use of the runner was advocated by William Nicholson in an article printed in the Philosophical transactions of 1787. But in the first half of the nineteenth century I have not been able to find a single reference to the "runner" in England. It was completely forgotten.

Returning to Newton, I shall take up my third point, the early use of the slide rule in the solution of numerical equations. Oldenburg's letter to Leibniz, previously referred to, reads in translation from the Latin as follows: "Mr. Newton, with the help of logarithms graduated upon scales by placing them parallel at equal distances or with the help of concentric circles graduated in the same way, finds the roots of equations. In the arrangement of these rules all the respective coefficients lie in the same straight line. From a point of which line, as far removed from the first rule as the graduated scales are from one another, in turn, a straight line is drawn over them, so as to agree with the conditions conforming with the nature of the equation; in one of these rules is given the pure power of the required root."

If my interpretation of this passage is correct, it means in the case of the cubic $x^3+ax^2+bx=c$ that the rules A, B, D must be placed parallel and equidistant. On rule A find the number equal to the numerical value of the coefficient a; on rule B find the number equal to the numerical value of b, and on rule D find unity. Then arrange these three numbers on the rules in a straight line BD. Select the point E on this line, so that $BE=BA$. Through E pass a line ED' and turn it about E until the numbers at B', A', and D', with their proper algebraic signs attached, are seen to be together equal to the absolute term c.



Then the number on the scale D' is equal to $|x^3|$, and x can be found.

Remembering that the length of $B''B$ is $\log |b|$, and assuming $BB'=\log |x|$, it follows that $B''B'$ is equal to $\log |bx|$. Then $AA'=2\log |x|$, or $\log |x^2|$ and $A''A'=\log |ax^2|$, and $DD'=\log |a^3|$. The value of x can be found by moving the scale B up until B'' reaches the point B. The number on the scale at B' will then give the numerical value of the root. A device, as represented by the line ED', fulfills some of the functions of what is now called the "runner."

In Stone's *Dictionary* (1743) Newton's scheme is modified somewhat. Stone assumes that the equation is so transformed that all its coefficients, except the absolute term, are positive, The rules are contiguous and are not all graduated alike, but have, respectively, a single, double, triple, quadruple, etc., radius. This device calls for a runner of the type now in use, carrying a thread that is at right angles to the rules. Otherwise the gen-

eral plan for the numerical solution of equations is the same as with Newton.

As a fourth point in the history of the slide rule, I desire to point out that, while so generally known to writers on the slide rule, the English astronomer William Pearson was the first one to suggest, in 1797, the inversion of the slider for certain operations with the slide rule, the inversion of fixed lines on the slide rule had been introduced more than one hundred years earlier in Everard's slide rule, used in gauging.

Finally, I desire to say a word as to the introduction of the slide rule into the United States. Brief directions for the use of the slide rule appeared in a few arithmetics imported, or reprinted in this country, in the latter part of the eighteenth century. Thus, the *Arithmetic* of George Fisher, which is a pseudonym for Mrs. Slack, probably the first woman who is the author of a popular arithmetic, contained rules for the use of the slide rule. Her books were read in the United States. In Nicolas Pike's arithmetic, an American text of 1788, such rules were given. An edition of the English book, Dilworth's *Schoolmaster's Assistant,* was brought out in Philadelphia in 1805 by Robert Patterson, professor of mathematics in the University of Pennsylvania. It devotes half a dozen pages to the use of the slide rule in gauging. Another English work, Hawney's *Complete Measurer* (1st English Edition, 1717), was printed in Baltimore in 1813. It describes the English carpenter's rule, also an English rule for gauging. Of American works, Bowditch's *Navigator,* 1802, gives one page to the explanation of the slide rule, but when working examples, Gunter's line alone is used. From these data it is difficult to draw reliable conclusions as to the extent to which the slide rule was then actually used in the United States. We surmise that it was practically unknown. The Swiss geodesist, F. R. Hassler, who came to this country and became the first superintendent of the United States Coast and Geodetic Survey, is known to have used a slide rule. The present writer had the good fortune of inspecting Hassler's slide rule. But before 1880 or 1885 it is very difficult to find references to the slide rule in American engineering literature. I have seen a reference to the slide rule in a book issued in the first half of the last century by a professor of the Rensselaer Polytechnic Institute. From this institute was graduated in 1863 Mr. Edwin Thacher, a bridge engineer, who in 1881 patented his well-known cylindrical slide rule. Interest in slide rules was awakened about this time. It was in 1881 that Robert Riddell published in Philadelphia his booklet on *The Slide Rule Simplified.* In the preface he points out that, though nearly unknown in this country, the instrument was invented before the time when William Penn founded Philadelphia. But the slide rule never became really popular in the United States until the introduction of the Mannheim type. Keuffel and Esser imported Mannheim rules in 1888 and began the manufacture of them in Jersey City in 1895. An inquiry* instituted by C. A.

**Engineering News,* Vol. 45, 1901, p. 405.

Holden in 1901 showed that in about half of the engineering schools of the United States, attention is given to the use of the slide rule.

A BIQUADRATIC EQUATION CONNECTED WITH THE REDUCTION OF A QUADRATIC LOCUS.

By DR. ARTHUR C. LUNN, The University of Chicago.

If the equation of a conic section be written in the form

$$Ax^2+By^2+2Cxy+2Dx+2Ey+F=0,$$

then it is known that a rotation of the coördinate axes through an angle a will bring them into parallelism with the axes of symmetry of the curve, provided this angle is determined by

$$\tan 2a=\frac{2C}{A-B}.$$

This rotation corresponds to the substitution

$$(1)\qquad \begin{aligned} x&=x'\cos a-y'\sin a,\\ y&=y'\cos a+x'\sin a, \end{aligned}$$

with a so chosen as to eliminate the term in $x'y'$. But the sine and cosine may be expressed in terms of the tangent of the half-angle, thus:

$$(2)\qquad t=\tan\frac{a}{2},\quad \cos a=\frac{1-t^2}{1+t^2},\quad \sin a=\frac{2t}{1+t^2},$$

and the use of these in (1) gives the substitution expressed rationally in terms of the parameter t. Without reference to its trigonometric source, the substitution in that form is seen to be orthogonal or rotational for all values of t, since the equation of constancy of distances:

$$(x'_1-x'_2)^2+(y'_1-y'_2)^2=(x_1-x_2)^2+(y_1-y_2)^2,$$

is directly verifiable as an identity in t.

The use of this parameter makes it possible to effect the reduction of the conic by purely algebraic processes, independently of the trigonometric formulae. For the term in $x'y'$ will have as coefficient

$$\frac{2}{(1+t^2)^2}\{2(B-A).t(1-t^2)+C[(1-t^2)^2-(2t)^2]\},$$

which will vanish if t satisfy the biquadratic equation:

$$(3) \qquad t^4+4mt^3-6t^2-4mt+1=0,$$

in which is put $m=\frac{A-B}{2C}$, the trigonometric value of which is $\cot 2a$. The four roots of this equation must therefore be real for all real values of m, and correspond to the four semi-axes of the conic.

But this equation must be solvable by quadratics. For, by a familiar formula of trigonometry

$$\tan 2a=\frac{2\tan a}{1-\tan^2 a}$$

which, regarded as a quadratic equation in $\tan a$ gives

$$\tan a=-\cot 2a \pm \surd[1+\cot^2 2a],$$

and a repetition of such solution gives $\tan\frac{1}{2}a$ in terms of $\cot a$. This suggests at once the four roots of the biquadratic in t, which after a little reduction prove to be:

$$\begin{array}{ll} t_1=-m+r+R_1 & \text{where} \\ t_2=-m-r+R_2 & r=\surd[1+m^2], \\ t_3=-m+r-R_1 & R_1=\surd[2(r^2-rm)], \\ t_4=-m-r-R_2 & R_2=\surd[2(r^2+rm)]. \end{array}$$

These roots are obviously all real, and are easily shown to be distinct. Direct computation shows that the product $(t-t_1)(t-t_2)(t-t_3)(t-t_4)$ gives the biquadratic polynomial on the left of the equation in t.

The reducibility of the biquadratic equation by quadratics is connected intimately with the existence of rational relations among the roots, which in the present case are the following:

$$t_1t_3=-1, \quad t_2t_4=-1, \quad \frac{t_2-t_1}{1+t_1t_2}=\frac{t_3-t_2}{1+t_3t_2},$$

and others (not independent) similar to the last. These correspond to the fact that, since the axes of the conic are mutually perpendicular, the various values of $\frac{1}{2}a$ must be spaced at intervals of 45°.

ON A CERTAIN CLASS OF QUARTIC CURVES.*

By PROF. R. D. CARMICHAEL, Anniston, Alabama.

The object of this paper is the discussion of a class of quartic curves whose equation may be represented thus: Set

$$(1) \qquad m_i = c_i \sqrt{[(x-a_i)^2+(y-b_i)^2]}, \quad i=1, 2, 3,$$

in which $c_i > 0$ and the radical is to be taken with the positive sign. Then put

$$(2) \qquad (m_1+m_2+m_3)(m_1+m_2-m_3)(m_1-m_2+m_3)(m_1-m_2-m_3)=0.$$

It is evident that if the factors of the first member of (2) are multiplied together the resulting equation is without radicals and is of the fourth degree in x, y. Therefore its locus is a quartic curve, except in the special case in which the reduced equation breaks up into rational factors.

It should be pointed out that in the equation in its expanded form each of the c's enters to an even power, and consequently the assumption of c_i positive is no limitation on the generality of equation (2). A similar remark applies to the assumption that the radicals are positive.

Attention will be confined to a discussion of the general nature of the locus and the problem of the construction of the curve by continuous motion when the c's are commensurable.

§1. General Nature of the Locus.

Equation (2) is evidently satisfied if any one of the factors of its first member is zero. But, since c_i and the radical in (1) each is to be taken positive, it is clear that

$$m_1+m_2+m_3=0$$

can be satisfied only when $x-a_i=0$ and $y-b_i=0$; that is, when $x=a_1=a_2=a_3$, $y=b_1=b_2=b_3$. The locus of (2) is simply a single point and is therefore not properly a locus of the fourth order. We shall therefore exclude the case from further consideration.

Each of the other three cases may be represented by

$$(3) \qquad m_\lambda = m_\mu + m_\nu, \qquad \lambda, \mu, \nu=1, 2, 3 \text{ in some order.}$$

It is evident at once from (1) that the branch which is represented by (3) has this property: If P is any point on the branch, then c_λ times its distance

*Read before the American Mathematical Society, October 26, 1907.

from (a_λ, b_λ) is equal to c_μ times its distance from (a_μ, b_μ) plus c_ν times its distance from (a_ν, b_ν). Since the c's are to be taken positive and the a's, b's, and c's are to be considered finite it is evident that this condition can always be satisfied whatever the positions of the three points and the values of the c's. Hence the locus of (2) has in general three branches, and these must be distinct except for special values.

Now suppose that any two branches, say,

$$m_\lambda = m_\mu + m_\nu , \quad m_\mu = m_\lambda + m_\nu ,$$

have some point in common. Then solving their equations as simultaneous, we have $m_\lambda = m_\mu$; and therefore each of the branches reduces to $m_\nu = 0$, an equation which can be satisfied only by the single point (a_ν, b_ν). That is, the two branches of the locus coincide and consist of a single point only. But, since $m_\lambda = m_\mu$, equation (2) now reduces to the form

$$(4) \qquad (2m_\lambda + m_\nu)(2m_\lambda - m_\nu)(m_\nu)^2 = 0,$$

the locus of which is evidently not properly of the fourth degree. The case, therefore, in which two branches of the locus of (2) may have a common point is to be excluded from further discussion.

We propose now to find a necessary condition for the existence of an infinite branch. Suppose that

$$(5) \qquad m_\lambda = m_\mu + m_\nu , \qquad \lambda, \mu, \nu = 1, 2, 3 \text{ in some order},$$

is such a branch. Then either x or y is infinite or both are infinite for some point of the locus. Now divide equation (5) by $\sqrt{[(x-a_\lambda)^2+(y-b_\lambda)^2]}$; in the result consider the case for which some point P is infinitely removed from the origin. The limit of each of the fractions of the form

$$\frac{\sqrt{[(x-a_\mu)^2+(y-b_\mu)^2]}}{\sqrt{[(x-a_\lambda)^2+(y-b_\lambda)^2]}}$$

is easily shown to be 1 when either x or y approaches infinity or when both approach infinity. Passing to the point P and taking the limiting values of the fractions in the equation which results from the suggested division in (5), we have

$$(6) \qquad c_\lambda = c_\mu + c_\nu ,$$

a condition which is necessary if the locus of (5) is an infinite branch. Then, evidently, *the locus of* (2) *can in no case have more than one infinite branch,*

and for the existence of such a branch it is necessary and sufficient that a relation (6) *shall hold.*

We consider now the relative position of the closed branches of the curve. Since no two branches can have a point in common, it follows that a closed branch cannot lie partly within and partly without another branch. Suppose that one of the closed branches lies entirely within another. Then draw a line through any point within the inner of these two ovals and through some point on the third branch (considered either as an oval or as an infinite branch). Such a line cuts the quartic curve in five points; and this is impossible. Therefore, *one of the closed branches cannot lie within the other.* A similar discussion leads to the theorem that no parts of two closed branches can lie on the same straight line with any point of the third branch. The facts of this paragraph will enable one to obtain an idea of the form of the locus in each of the two possible cases which may arise.

§2. Construction by Continuous Motion.

In this section the discussion is confined to the case in which the c's are commensurable.

Since the equation of any branch may be written in the form

$$m_\lambda = m_\mu + m_\nu\,, \tag{7}$$

it follows that the problem of construction by continuous motion is completely solved when any branch in its most general case has been constructed.

Since the c's are now to be considered commensurable, we may multiply them by some common number d so that the resulting numbers are integers. Then let $dc_i=k_i$, k_i an integer. Consider the construction of the branch whose equation is of the form

$$dm_1=dm_2+dm_3. \tag{8}$$

The coefficient of each radical in the equation is now an integer. The radicals represent the distance of a point P on the locus from A, B, and C, respectively; A, B, C being in order the points (a_1, b_1), (a_2, b_2), (a_3, b_3). Let smooth pegs be placed at A, B, and C.

Now place a pencil at P. Take a cord of convenient length and pass it around the pencil at P and the peg at A and attach it either to the pencil point or to the peg so that $dc_1=k_1$ plies extend from P to A. (The cord is attached to the pencil if k_1 is even; to the peg, if k_1 is odd.) The unattached end passes out by the pencil at P in a convenient direction.

Take a second cord having the free end in the same direction as the free end of the first cord. Pass the other end around the pencil at P and the peg at B, leaving it yet unattached, so that $dc_2=k_2$ plies extend from P

to B. If k_2 is even the cord passed last to the pencil; then pass it around the point C making k_3 plies between P and C, and attach the end to the pencil or to the peg at C according as k_3 is even or odd. But if k_2 is odd, the cord passed last to the peg at B; then let it pass from B to C, and then from C around the pencil at P until the requisite number, k_3, of plies extend from C to P; finally attach the end to the pencil or to the peg at C according as k_3 is odd or even.

Now let both cords be stretched tight while the pencil is held firmly at P. Then tie the free ends of the cords together at some convenient distance from the pencil so that when a pull is made on the knot both strings will be drawn tight throughout their entire lengths, with the exception of course of the free ends beyond the knot. Then if the pencil moves and the cords are kept always in the position which has been defined, it is evident that the pencil point describes the branch in consideration; for k_1 times the distance from A remains always equal to k_2 times the distance from B plus k_3 times the distance from C.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

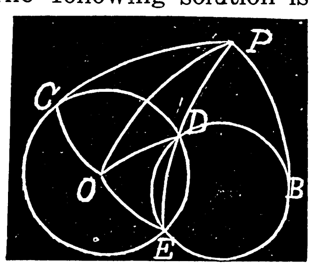
GEOMETRY.

20. Proposed by **DR. GEORGE BRUCE HALSTED**, Greeley, Colo.

Demonstrate by pure spherical geometry that spherical tangents from any point in the produced spherical chord common to two intersecting circles on a sphere are equal.

Solution by **G. B. M. ZERR, A. M., Ph. D.**, 4243 Girard Avenue, Philadelphia, Pa.

No solution of this problem has yet appeared in the MONTHLY. A simple geometrical solution such as is possible for the corresponding problem in planes is not possible for this problem. The following solution is quite simple.



Let P be point on the common chord DE; PB, PC the tangents, O the pole of one circle. Let $PE=R$, $PD=r$, $PC=\rho$, $PB=\rho'$, $PO=\delta$, $OD=OE=OC=\beta$, $\angle EPO=\phi$.

Then $\cos\beta=\cos R\cos\delta+\sin R\sin\delta\cos\phi\ldots(1)$,

$\cos\beta=\cos r\cos\delta+\sin r\sin\delta\cos\phi\ldots(2)$,

$\cos\delta=\cos\beta\cos\rho\ldots(3)$,

$\cos\phi$ from (1) in (2) gives $\cos\beta(\sin R-\sin r)=\cos\delta\sin(R-r)\ldots(4)$.

$\cos\delta$ from (3) in (4) gives $\cos\rho=(\sin R-\sin r)/\sin(R-r)$.

Similarly, $\cos\rho'=(\sin R-\sin r)/\sin(R-r)$. $\therefore \rho=\rho'$.

These equations reduce to $\tan^2\frac{1}{2}\rho=\tan^2\frac{1}{2}\rho'=\tan\frac{1}{2}R\tan\frac{1}{2}r$.

Professor Philbrick gave a solution of this problem at the time it was published but it did not fill the requirements because it was not a pure spherical geometry solution.

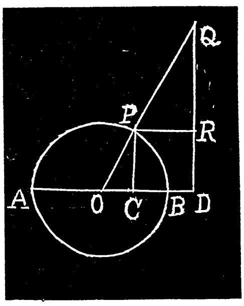
290. Proposed by J. J. QUINN, Scottdale, Pa.

(a) Suppose a circle described around the origin. Then at the end of a uniformly revolving radius r, a line equal to the diameter is pivoted. Find the equation of the locus of its extremity, if for every unit of angle its projection on the X axis is a constant linear unit, being the same part of the diameter as the angle is of π radians.

(b) Show how it can be applied to the trisection or multisection of an angle.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

(a) Let angle $POB=\theta$. Then CD, the projection of $PQ=2r$ on AB, is $r\,\theta/90$. $OD=x=r\cos\theta+r\,\theta/90$.



$DQ=y=r\sin\theta+r\sqrt{[4-(\theta/90)^2]}$.

$\rho^2=x^2+y^2=5r^2+2r^2\cos\theta(\theta/90)$ $+2r^2\sin\theta\sqrt{[4-(\theta/90)^2]}$ is the polar equation sought.

(b) Let $m\phi$ be the angle to be multisected. $m\phi/m=\phi$. Lay off $OD=x=r\cos\phi+r\,\phi/90$. Then erect $DQ=y=r\sin\phi+r\sqrt{[4-(\phi/90)^2]}$ perpendicular to OD at D. From Q as center, with radius equal to $2r$ describe an arc cutting the circumference of the given circle at P.

Draw PO; then $\angle POD=\phi$.

300. Proposed by J. J. QUINN, Ph. D., Scottdale, Pa.

Trisect an angle by means of a tractrix.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philahelphia, Pa.

The length of the tangent between the axis of abscissas and the point of tangency is constant.

Let a=length of this tangent, y=an ordinate opposite angle θ, z=an ordinate opposite angle ϕ. Also let $\theta=3\phi$.

$\therefore y=a\sin\theta=3a\sin\phi-4a\sin^3\phi$, $z=a\sin\phi$.

$\therefore y/z=3-4\sin^2\phi$ or $\sin\phi=\frac{1}{2}\sqrt{[(3z-y)/z]}$.

$\therefore z^2/a^2=\dfrac{3z-y}{4z}$ or $y=\dfrac{(3a^2-4z^2)z}{a^2}$.

Let $PD=z$, $PCD=\angle\phi$. Construct $QB=y=\dfrac{(3a^2-4z^2)z}{a^2}$.

Let $QAB=\theta$ where $PC=QA=a$. Then $\theta=3\phi$.

$\therefore$ Parallel to PC draw AR, then $\angle RAB=\frac{1}{3}\angle QAB$.

$PD=z$ cannot be greater than $\frac{1}{2}a$, then $y=a$, $\theta=\frac{1}{2}\pi$, $\phi=\frac{1}{6}\pi$.

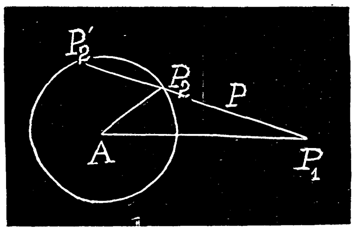
319. Proposed by S. F. NORRIS, Baltimore City College, Baltimore, Md.

Lines are drawn from a fixed point P_1, meeting a fixed circle in P_2. On P_1P_2 a point P is taken so that $P_1P \times P_1P_2 = k^2$. Find the locus of P. Solve by analytic methods, using rectangular coordinates, and putting the result in the form,

$$(x_1^2+y_1^2-r^2)[(x-x_1)^2+(y-y_1)^2]+2k^2(x_1x+y_1y-x_1^2-y_1^2)+k^4=0.$$

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let AP_1 be the axis of abscissas, A the origin of orthogonal coordinates, x and y the co-ordinates of P, ϕ the $\angle P_2P_1A$, and θ the $\angle AP_2P'_2$. Then $PP_1 = \dfrac{y}{\sin\phi}$, $P_1P_2 = x_1\cos\phi - r\cos\theta$. Therefore from the condition $PP_1 \times P_1P_2 = k^2$ we have $\dfrac{y}{\sin\phi}(x_2\cos\phi - r\cos\theta) = k^2$, but $\tan\phi = \dfrac{y}{x_1-x}$. $\therefore \sin\phi = \dfrac{y}{\sqrt{[y^2+(x_1-x)^2]}}$, $\cos\phi = \dfrac{x_1-x}{\sqrt{[y^2+(x_1-x)^2]}}$, $\sin\theta = \dfrac{x_1\sin\phi}{r}$. Substituting, we get $\dfrac{y}{\sin\phi}[x_1\cos\phi - \sqrt{(r^2-x_1^2\sin^2\phi)}\,[=k^2$, or, removing the radical, $x_1^2y^2 - 2k^2x_1\sin\phi\cos\phi\, y + k^4\sin^2\phi = r^2y^2$, and substituting now the values for $\sin\phi$ and $\cos\phi$ we obtain, after some reductions that admit of no difficulty,



$$y^2+\left[x+\frac{k^2+r^2-x_1^2}{x_1^2-r^2}x_1\right]^2=\frac{k^4r^2}{(x_1^2-r^2)^2},$$

or putting $x_1^2-r^2=a^2$=square of the tangent drawn from P_1,

$$y^2+\left[x+\frac{k^2-a^2}{a^2}x_1\right]^2=\frac{k^4r^2}{a^4},$$

the equation of a circle, whose center lies on AP_1, at a distance $=\dfrac{a^2-k^2}{a^2}x_1$ from A, and whose radius$=\dfrac{k^2r}{a^2}$.

Also solved by G. B. M. Zerr.

CALCULUS.

247. Proposed by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Integrate, $x\dfrac{\partial^2y}{\partial x^2}+2\dfrac{\partial y}{\partial x}-xy=0.$

Solution by GEORGE W. HARTWELL, Columbia University, New York.

Dividing by x, $\frac{\partial^2 y}{\partial x^2}+\frac{2}{x}\frac{\partial y}{\partial x}-y=0\ldots(1)$.

Remove the term containing $\frac{\partial y}{\partial x}$ by substituting $y=\frac{v}{x}$. Then (1) becomes $\frac{\partial^2 v}{\partial x^2}-v=0\ldots(2)$.

The solution of this equation is $v=c_1e^x+c_2e^{-x}$. $\therefore xy=(c_1e^x+c_2e^{-x})$.

Solved in a similar manner by G. B. M. Zerr.

248. Proposed by REV. R. D. CARMICHAEL, Anniston, Ala.

Evaluate $\int_0^{\frac{1}{2}\pi}\sin nx\cot x dx$, where n is a positive integer.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

$$A=\int_0^{\frac{1}{2}\pi}\sin nx\ \cot x\ dx=\int_0^{\frac{1}{2}\pi}\frac{\sin(n-1)xdx}{\sin x}+\int_0^{\frac{1}{2}\pi}\cos nx\ dx$$

$$=2\int_0^{\frac{1}{2}\pi}[\cos x+\cos 3x+\cos 5x+\ldots+\cos(n-3)x]dx+\int_0^{\frac{1}{2}\pi}\cos nx\ dx$$

$$=2\Big[\sin x+\tfrac{1}{3}\sin 3x+\tfrac{1}{5}\sin 5x+\ldots+\frac{1}{n-3}\sin\ (n-3)\ x\Big]_0^{\frac{1}{2}\pi}+\frac{1}{n}\cos\frac{\pi n}{2}$$

$$=2\Big[1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\ldots\pm\frac{1}{n-3}\Big]\pm\frac{1}{n};\text{ for } n \text{ even}=\tfrac{1}{2}\pi \text{ if } n=\infty.$$

$$A=\int_0^{\frac{1}{2}\pi}[1+2\cos 2x+2\cos 4x+2\cos 6x+\ldots+2\cos(n-3)x]dx$$

$$+\int_0^{\frac{1}{2}\pi}\cos nx\ dx=\Big[x+\sin 2x+\tfrac{1}{2}\sin 4x+\tfrac{1}{3}\sin 6x+\ldots+\frac{2}{n-3}\sin(n-3)x\Big]_0^{\frac{1}{2}\pi}$$

$$+\frac{1}{n}\cos\frac{\pi n}{2}=\tfrac{1}{2}\pi,\text{ for } n \text{ odd.}$$

249. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Ike, running with constant velocity v, is trying to catch Jim, running with constant velocity V, $(V>v)$, by keeping Jim dead ahead of him. Find their paths.

Solution by the PROPOSER.

Let I, J be the positions of Ike and Jim at the start; L, M their respective positions at any subsequent time t.

Let m, n be the coordinates of L; p, q the coordinates of M; ϕ the angle LM makes with IJ, $a=IJ$.

Then $dm/dt=v\cos\phi$, $dn/dt=v\sin\phi$, $p=m+a\cos\phi$, $q=n+a\sin\phi$.

$V^2=(dp/dt)^2+(dq/dt)^2$

$=(dm/dt-a\sin\phi\, d\phi/dt)^2+(dn/dt+a\cos\phi\, d\phi/dt)^2$

$=(v\cos\phi-a\sin\phi\, d\phi/dt)^2+(v\sin\phi+a\cos\phi\, d\phi/dt)^2$

$=v^2+a^2(d\phi/dt)^2$; $\therefore d\phi/dt=\sqrt{(V^2-v^2)}/a=b$.

$\therefore \phi=bt$, since $\phi=0$, when $t=0$.

$\therefore dm/dt=v\cos bt$, where $b=\sqrt{(V^2-v^2)}/a$.

$\therefore m=-\dfrac{v}{b}\sin bt$, $n=\dfrac{v}{b}(\cos bt-1)$. Therefore, Ike describes a circle.

Also, $p=a\cos bt-(v/b)\sin bt$, $q=a\sin bt+(v/b)(\cos bt-1)$.

Therefore, Jim also describes a circle.

MECHANICS.

191. Proposed by DR. L. E. DICKSON, The University of Chicago.

Give the axiomatic principle of Physics which is equivalent to the theorem on the compound of two circles ("Graphical Methods in Trigonometry," MONTHLY, June-July, 1905).

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

There are two principles that might be considered equivalent to the theorem on the compound of two circles. First, the parallelogram of velocities; second, the parallelogram of forces.

These might be named the compound of two velocities and the compound of two forces. We can state both under one theorem as follows:

The compound of a $\left\{\begin{matrix}\text{velocity}\\\text{force}\end{matrix}\right\}$ OP with a $\left\{\begin{matrix}\text{velocity}\\\text{force}\end{matrix}\right\}$ OR is the diagonal OQ of the parallelogram $OPQR$.

The proof by vectors follows at once. Regarding OP, OR, OQ as vectors we get at once $OP+OR=OQ$.

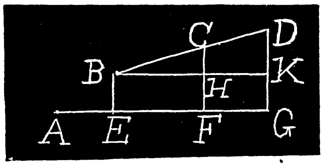
207. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

A portion of a parabola is bounded by the curve, the axis and an ordinate. A circle is inscribed to the figure which is regarded as a plane lamina. The area of the inscribed circle is now punched out. Find the centroid of what is left.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philahelphia, Pa.

Let $y^2=4ax$ be the parabola, b the abscissa of the portion considered,

r the radius of the circle. Then $(x-b+r)^2+(y-r)^2=r^2$ is the equation to the circle. Hence the invariants are $\Delta=-4a^2$, $\Delta'=-r^2$, $\theta=-4a(b-r+a)$, $\theta'=-4a(b-r)$. In order that the circle and parabola may touch $(\theta\theta'-9\Delta\Delta')^2=4(\theta^2-3\Delta\theta')\times(\theta'^2-3\Delta'\theta)$.



$\therefore 16r^5+(13a-48b)r^4+8(a^2+6b^2-14ab)r^3+8(15ab^2-2b^3-9a^2b)r^2+32ab(3ab-a^2-2b^2)r+16ab^2(b^2+a^2-2ab)=0.$ This gives the value of r.

Area of semi-parabola$=\frac{4}{3}b\sqrt{(ab)}$, area of circle$=\pi r^2$, area of portion left$=\frac{4}{3}b\sqrt{(ab)}-\pi r^2$.

Let A be the vertex of the parabola, B the centroid required, C the centroid of the semi-parabola, D the centroid of the circle. Let (α, β) be the coordinates of B. The coordinates of C are $(\frac{3}{5}b, \frac{3}{4}\sqrt{ab})$; of D, $(b-r, r)$.

$$\frac{CD}{BC}=\frac{HK}{BH}=\frac{GF}{EF}=\frac{\frac{4}{3}b\sqrt{(ab)}-\pi r^2}{\pi r^2} \text{ or } \frac{EG}{EF}=\frac{4b\sqrt{(ab)}}{3\pi r^2}.$$

$$\therefore \frac{b-r-\alpha}{\frac{3}{5}b-\alpha}=\frac{4b\sqrt{(ab)}}{3\pi r^2}. \quad \therefore \alpha=\frac{3[4b^2\sqrt{(ab)}+5\pi r^3-5\pi br^2]}{5[4b\sqrt{(ab)}-3\pi r^2]}.$$

$$\text{And } \frac{DK}{CH}=\frac{EG}{EF}=\frac{4b\sqrt{(ab)}}{3\pi r^2}=\frac{r-\beta}{\frac{3}{4}\sqrt{(ab)}-\beta}; \quad \therefore \beta=\frac{3(ab^2-\pi r^3)}{4b\sqrt{(ab)}-3\pi r^2}.$$

208. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, Eng.

Hanging at rest over a smooth pulley are two equal scale pans of the same mass. Two equal particles, the one inelastic and the other elastic, are simultaneously dropped from the same height one into each scale pan. Show that each impact after the first must occur when the pans have returned to the *status quo ante*, and find the total space described by either pan before motion ceases.

Remark by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

This is the same as problem 121, Mechanics. A solution of this problem is found in Vol. VIII, No. 10, pp. 203-4, October, 1901.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

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141. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Given that the highest factor of a prime p contained in $m!$ is p^{m-s}; find general expressions involving p and m and s, from which, when a solution is possible, m can be determined when s is a given integer and p is a given prime. Is it then possible in any case to have more solutions than one?

Solution by the PROPOSER.

The result takes different forms according as p is 2 or is an odd prime.

First suppose $p=2$. According to Legendre (Théorie des nombres, 3rd ed., I., p. 10) the highest power of 2 contained in

$$m=2^{\alpha}+2^{\beta}+2^{\gamma}+\dots\text{to } s \text{ terms, } \alpha,\ \beta,\ \gamma,\ \dots \text{ different integers, is } 2^{m-s};$$

therefore when $p=2$, there is an infinite number of solutions. Every m which may be expressed as a sum of s different powers of 2 will satisfy the imposed conditions.

Henceforth take p an odd prime. Suppose m expressed in the form

$$m=c_n p^n+c_{n-1}p^{n-1}+\dots+c_1 p+c_0, \tag{1}$$

where each c is positive and less than p, or is zero. Then in any case m is determined when the values of the c's are found.

According to Legendre (l. c.) the highest power of p contained in m is that of which the index is $\dfrac{m-(c_n+c_{n-1}+\dots+c_1+c_0)}{p-1}$. But, this is the $m-s$ of the problem. Therefore,

$$m-s=\frac{m-(c_n+c_{n-1}+\dots+c_1+c_0)}{p-1}.$$

This equation reduces to

$$m+\frac{c_n+c_{n-1}+\dots+c_1+c_0}{p-2}=\left(1+\frac{1}{p-2}\right)s;$$

or, replacing m by its value from (1), we have

$$c_n p^n+c_{n-1}p^{n-1}+\dots+c_1 p+c_0+\frac{c_n+c_{n-1}+\dots+c_1+c_0}{p-2}=\left(1+\frac{1}{p-2}\right)s. \tag{2}$$

Now, the second member of this equation is a known quantity. If it is written in the form

$$\left(1+\frac{1}{p-2}\right)s=r_v p^v+r_{v-1}p^{v-1}+\dots+r_1 p+r_0+\frac{n}{p-2},$$

where each r is positive and less than p or is zero, and $u<p-2$, we have, from (2),

$$(3)\qquad c_n p^n+c_{n-1} p^{n-1}+\ldots+c_1 p+c_0+\frac{c_n+c_{n-1}+\ldots+c_1+c_0}{p-2}$$
$$=r_v p^v+r_{v-1} p^{v-1}+\ldots+r_1 p+r_0+\frac{n}{p-2},$$

from which the values of the c's (and hence by (1) the value or values of m) may be found.

As an example, take the following case: $p=5$ and $s=3725$. We have from (3),

$$c_n p^n+c_{n-1} p^{n-1}+\ldots+c_1 p+c_0+\frac{c_n+c_{n-1}+\ldots+c_1+c_0}{p-2}$$
$$=5^5+2.5^4+4.5^3+3.5^2+3.5+1+\tfrac{2}{3}.$$

It is easy to show that $n=5$ and $c_5=1$; then that $c_4=2$, and $c_3=4$, and $c_2=3$, and $c_1=2$, and $c_0=2$. (Notice that the values of the c's are most easily determined in the order given.) This gives

$$m=5^5+2.5^4+4.5^3+3.5^2+2.5+2=4962.$$

It is now evident that the solution or solutions may *always* be readily obtained from (3).

When $s=5$ and $p=3$, $m=7$ or 9; and this is sufficient to show that more than one solution exists in some cases even when p is odd.

An incomplete solution was received from Professor Zerr.

143. Proposed by JOHN D. WILLIAMS (being the first of 14 challenge problems published in 1832).

Make $x^2+y^2=a^2=z^2+w^2$ and $x^2-w^2=z^2-y^2=\square$.

I. Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let $x=2q(7q-5p)$, $y=25p^2-70pq+48q^2$,
$z=24p^2-70pq+50q^2$, $w=p(10q-7p)$.
$\therefore x^2+y^2=z^2+w^2=(25p^2-70pq+50q^2)^2=a^2$.
$x^2-w^2=z^2-y^2=196q^4-280pq^3+140p^3q-49p^4=(14q^2-10pq-\frac{25}{7}p^2)^2$
when $q=\dfrac{1513p}{1680}$; $m=\frac{1513}{1680}$, where $pm=q$.

$$\therefore x=\frac{473569p^2}{201600},\ y=\frac{52319p^2}{58800},\ z=\frac{85345p^2}{56448},\ w=\frac{337p^2}{168}.$$

Reducing to a common denominator and expunging common factors, we get

$x=3314983p^2$, $y=1255656p^2$, $z=2133625p^2$, $w=2830800p^2$.
$\therefore x^2+y^2=z^2+w^2=a^2=(3544825p^2)^2$,
$x^2-w^2=z^2-y^2=b^2=(1725017p^2)^2$.
Also $a-x=2(339p)^2=2c^2$, $a-z=2(840p)^2=2d^2$,
$a-y=(1513p)^2=h^2$, $a-w=(845p)^2=k^2$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philahelphia, Pa.

General expressions for the sides of duplicate right-triangles having the same hypotenuse are

$x=(p^2-q^2)(r^2-s^2)+4pqrs,\ y=2rs(p^2-q^2)-2pq(r^2-s^2),$

$z=(r^2-s^2)(p^2+q^2),\ w=2rs(p^2+q^2).$

$\therefore x^2+y^2=z^2+w^2=[(r^2+s^2)(p^2+q^2)]^2$, Let $p=2,\ q=1$.

$\therefore x=3(r^2-s^2)+8rs,\ y=6rs-4(r^2-s^2),\ z=5(r^2-s^2),\ w=10rs.$

$x^2+y^2=z^2+w^2=[5(r^2+s^2)]^2,\ x^2-w^2=z^2-y^2=9r^4+48r^3s-54r^2s^2-48rs^3+9s^4=\square$. This is a square when $r=\frac{17}{12}s$.

$$\therefore x=\frac{689s^2}{64},\ y=\frac{161s^2}{36},\ z=\frac{725s^2}{144},\ w=\frac{85s^2}{6}.$$

$\therefore x=2067s^2,\ y=644s^2,\ z=725s^2,\ w=2040s^2.$

$x^2+y^2=z^2+w^2=(2165s^2)^2=a^2,\ x^2-w^2=z^2-y^2=(333s^2)^2=b^2.$

A solution of this problem is given in J. D. Williams' *Algebra*, page 419. He starts with $a^2=b^2+f^2=c^2+e^2$, and $b^2-c^2=d^2=e^2-f^2$. Then he is to make a^2-b^2 a square, a^2-c^2 a square, and b^2-f^2 a square. He assumes $a^2=(p^2+q^2)(r^2+s^2)$, $b=pr\pm qs$, $c=ps\pm qr$. Then he assumes $r=pm-qn$, $s=pn+qm$. He finally arrives at the conclusion that $a=697$, $b=680$, $f=153$, $c=672$, $e=185$, a set of erroneous values, as Dr. Zerr has pointed out. It is likely that Williams' solution may be carried out so that a set of correct values may be obtained. Williams proposed this problem in 1832 as a challenge problem to the mathematicians of the United States. ED. F.

144. Proposed by JOHN D. WILLIAMS (being the ninth of his 14 challenge problems proposed in 1832).

Make $(m^2+n^2)^2x^2\pm(m^2+n^2)x=\square$, $(m^2-n^2)^2x^2\pm(m^2-n^2)x=\square$, and $4m^2n^2x^2\pm2mnx=\square$.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let $m^2+n^2=p,\ m^2-n^2=q,\ 2mn=r$.

$\therefore p^2x^2\pm px=\square,\ q^2x^2\pm qx=\square,\ r^2x^2\pm rx=\square\ldots(1,\ 2,\ 3).$

Let $p^2x^2+px=a^2x^2$; $\therefore x=p/(a^2-p^2)$.

This value of x in (2) and (3) gives

$$q^2p^2+qp(a^2-p^2)=\square,\quad r^2p^2+rp(a^2-p^2)=\square\ldots(4,\ 5).$$

Let $q^2p^2+qp(a^2-p^2)=[pq-b(a-p)]^2$.

$\therefore (a-p)=\dfrac{2pq(b+p)}{b^2-pq}$. This value of a in (5) gives

$$r^2b^4+4b^3pqr+2b^2pqr(2p+2q-r)+4bp^2q^2r+p^2q^2r^2=\square$$
$$=(rb+2bpq-pqr)^2,\text{ suppose.}$$

$$\therefore b=\frac{2pqr}{pr+qr-pq};\ x=\frac{p}{a^2-p^2}=\frac{(b^2-pq)^2}{4bpq\ (b+p)\ (b+q)}.$$

$$\therefore x=\frac{-[(pr+qr-pq)^2-4pqr^2]^2}{8pqr(pr+qr-pq)(pq-pr+qr)(pq+pr-qr)}.$$

$$\therefore x = \frac{-[(4m^3n-m^4+n^4)^2-16m^2n^2(m^4-n^4)]^2}{16mn(m^4-n^4)(4m^3n-m^4+n^4)(m^4-n^4-4mn^3)(m^4-n^4+4mn^3)}$$

$$= \pm\frac{A^2}{16mn\,(m^4-n^4)B}, \text{ suppose.}$$

x is $\pm$ according as px, qx, rx is $\mp$.

$\therefore (m^2+n^2)x^2 \pm (m^2+n^2)x$

$$=\left[\frac{A}{16mn\,(m^2-n^2)B}\right]^2[16m^2n^4(n^2-2m^2)+(8mn^3+n^4-m^4)(m^4-n^4)]^2.\ (6).$$

$(m^2-n^2)x^2 \pm (m^2-n^2)x$

$$=\left[\frac{A}{16mn\,(m^2+n^2)B}\right]^2[16m^2n^4(n^2+2m^2)-(8mn^3+m^4-n^4)(m^4-n^4)]^2.\ (7).$$

$4m^2n^2x^2 \pm 2mnx$

$$=\left[\frac{A}{8(m^4-n^4)\,B}\right][3(m^4-n^4)^2-8m^3n(m^4-n^4)-16m^2n^6]^2.\ (8).$$

m and n can have any values that make B positive. Let $m=2$, $n=1$; $A=671$, $B=2737$.

(6) gives $(m^2+n^2)^2x^2 \pm (m^2+n^2)x=(290543/262752)^2$.

(7) gives $(m^2-n^2)^2x^2 \pm (m^2-n^2)x=(74481/437920)^2$.

(8) gives $4m^2n^2x^2) \pm 2mnx=(234179/328440)^2$.

AVERAGE AND PROBABILITY.

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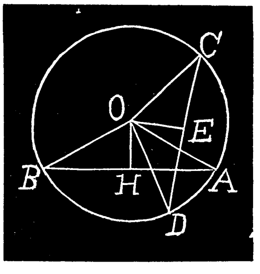
191. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Two random lines cut a given circle. What is the chance that they intersect within the circle?



II. Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let AB, CD be the random lines, $\angle AOH=\theta$, $\angle COE=\phi$, $\angle EOH=\psi$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ, 0 and θ; of ψ, $\theta-\phi$ and $\theta+\phi$ for favorable cases, and 0 and π for total cases.

Hence the chance is

$$p=\frac{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\theta-\phi}^{\theta+\phi}d\,\theta\,d\,\phi\,d\,\psi}{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{\pi}d\,\theta\,d\,\phi\,d\,\psi}$$

$$=\frac{8}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\theta-\phi}^{\theta+\phi}d\,\theta\,d\,\phi\,d\,\psi=\frac{16}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\phi\,d\,\theta\,d\,\phi=\frac{8}{\pi^3}\int_0^{\frac{1}{2}\pi}\theta^2 d\,\theta=\tfrac{1}{3}.$$

One of our readers has again called up the question of the correctness of the two solutions, saying that he is unable to decide which is correct. We shall continue to repeat onr answer as long as this question is asked. There is no such thing as *the correct solution* of the two published solutions—one is just as correct as the other. Both are correct when the law of distribution of the events assumed are granted. To these two solutions might be added an indefinite number of other solutions of equal merit. The fact of the matter is, that the problem is stated in the *indefinite* form, and when thus stated admits of an indefinite number of solutions. There is nothing to prevent one from assuming that the lines are drawn through each of two points taken at random within the circle, making it the equivalent of problem 5450 of the *Educational Times*, as was referred to by our reader. Were one to add to the problem referred to in the *Educational Times* the law of distribution of the points, then this problem would be definite, and there could be only one correct solution possible. ED. F.

MISCELLANEOUS.

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169. Proposed by E. D. ROE, Ph. D., Syracuse University, Syracuse, N. Y.

Find the value for all finite values of k of

$$\operatorname*{Lim.}_{x\doteq\infty}\left[x^k\log\left(\frac{e^x+1}{e^x-1}\right)\right].$$

Solution by the PROPOSER.

1. If x is positive, it will be useful to write,

$$x^k\log\left[\frac{e^x+1}{e^x-1}\right]=e^{-x}e^x x^k\log\left[\frac{1+e^{-x}}{1-e^{-x}}\right]=\frac{x^k}{e^x}\log\left[\left(1+\frac{1}{e^x}\right)^{e^x}\left(1-\frac{1}{e^x}\right)^{-e^x}\right].$$

Therefore, $\underset{x\doteq\infty}{L}\,x^k\log\left[\frac{e^x+1}{e^x-1}\right]=\log e^2.\underset{x\doteq\infty}{L}\frac{x^k}{e^x}=2L\frac{x^k}{e^x}.$

Since $\frac{x^k}{e^x}=\frac{x^k}{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots}$, $\underset{x\doteq\infty}{L}\frac{x^k}{e^x}=0$, for all finite values of k, and

therefore $\underset{x\doteq\infty}{L}\,x^k\log\left[\frac{e^x+1}{e^x-1}\right]=0.$

2. If x is negative, $\underset{x\doteq\infty}{L}\,x^k\log\left[\frac{e^x+1}{e^x-1}\right]=\underset{x\doteq\infty}{L}\,x^k.\log(-1)=\infty.$

Also solved by G. B. M. Zerr, J. Scheffer, and S. A. Corey.

PROBLEMS FOR SOLUTION.

ALGEBRA.

295. Proposed by CHARLES GILPIN, JR., Philadelphia, Pa.

In the equation $x^3-ax\pm b=0$, we have the following relation between the coefficients and the roots: (1) When $a^3/b^2=6.75$ there are three real roots, two of which are equal; (2) when $a^3/b^2<6\cdot75$ there are two imaginary roots and one real one; and (3) when $a^3/b^2>6.75$ there are three real, unequal roots.

296. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philahelphia, Pa.

Sum the series, $1+\frac{1}{6}+\frac{1}{20}+\frac{1}{50}+\frac{1}{105}+\frac{1}{196}+\frac{1}{336}+\ldots$

GEOMETRY.

329. Proposed by JOHN JAMES QUINN, Ph. D., Scottdale, Pa.

1. Determine the equation of the locus of a fixed point in a circle of radius r, rolling along the axis of an upright cylinder of the same radius, while the axis revolves (carrying the circle with it) through an angle equal to the central angle of the rolling circle formed by the radii to the fixed point and the point of contact.
2. Suppose the point projected into the surface of the cylinder.
3. What is the surface generated by the radius of the rolling circle?
4. What is the surface generated by a radius of the cylinder through the moving point?

CALCULUS.

252. Proposed by J. H. MEYER, S. J., Augusta, Ga.

Supposing the arc of a semi-circle to be stretched out into a straight line, and an indefinite number of perpendiculars erected on it, each equal to the versed sine of the corresponding arc; what would be the length of the curve traced out by the tops of the perpendiculars?

253. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Find the maximum number of real points of inflection in each of the quartic curves $y^2=\alpha x^4\pm x^2+\beta$, and find the necessary and sufficient relations between α and β for the existence of this number of points of inflection.

MECHANICS.

212. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, Eng.

A peg A is vertically d feet above a peg B. A string AD, a feet long, with two equal, jointed rods DC, CB form the whole figure. Discuss the position of equilibrium.

213. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

Two unequal, uniform, smoothly hinged rods are placed over a smooth vertical circle. Apply the principle of vertical work to find the condition of equilibrium in terms of the length of each rod, the diameter of the circle and the angle of either rod with the vertical.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

151. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

In the recurring series, $n=0, 1, 2, 3, 4, 5, 6, 7, \ldots$
$u_n=3, 0, 2, 3, 2, 5, 5, 7, \ldots$
where the scale of relation is $u_{n+3}=u_{n+1}+u_n$, prove that u_p is always divisible by p when p is prime. Is the converse true?

AVERAGE AND PROBABILITY.

193. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

What is the average area of all squares that may be inscribed in a given sector of a circle, a diagonal of the square being parallel to a random line across the sector?

194. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

What is the mean value of the triangle formed by joining three points taken at random on the circumference of a circle?

MISCELLANEOUS.

176. Proposed by WM. E. HEAL, Coffeyville, Kansas.

In Grassman's *Extensive Algebra*, $e_1e_2=-e_2e_1$. If $e_1=e_2$, $e_1^2=-e_1^2=0$. In quaternions, $ij=-ji$, $i^2j=i.ij=ik=-j$, $i^2=-1$. Reconcile these apparently divergent results.

NOTES AND NEWS.

The next Summer meeting of the American Mathematical Society will be held at the University of Illinois during September.

On December 17th occurred the death of Lord Kelvin, one of the greatest mathematicians and physicists of the present age.

On the 7th of November, 1907, occurred the death of Professor J. R. Rand, Professor of Mathematics in Bates College, Lewiston, Me. His successor is George E. Ramsdell.

During the last convocation week, Professor G. A. Miller was elected vice president of the American Mathematical Society, chairman of the Chicago Section of this Society, and secretary of Section A of the American Association for the Advancement of Science. The last position is for five years, while the other two are for one year only.

BOOKS.

Irrational Numbers and their Representation by Sequences and Series. By Henry Parker Manning, Ph. D., Assistant Professor of Pure Mathematics in Brown University. 12mo. Cloth, vi+123 pages. Price, $1.25. New York: John Wiley & Sons.

In this book, the author has given an explanation of irrational numbers and those parts of algebra which depend on the Theory of Limits. An *irrational number* is defined by the position it occupies among rational numbers, the definition given by Dedekind.

The book contains five chapters, of which the first deals with Irrational Numbers; the second, Sequences; the third, Series; the fourth, Power Series; and the fifth, the Exponential, Binomial, and Logarithmic Series.

The work will prove very helpful to all teachers of mathematics who wish to know something of the modern views of irrational numbers and series. F.

Introduction to Higher Algebra. By Maxime Bôcher, Professor of Mathematics in Harvard University. Prepared for publication with the co-operation of E. P. R. Duval, Instructor in Mathematics in the University of Wisconsin. 8vo. Cloth, xi+321 pages. Price, $2.00. New York: The Macmillan Co.

The object of this book is to "introduce the student to higher algebra in such a way that he shall, on the one hand, learn what is meant by a proof in algebra and acquaint himself with the proofs of the most fundamental facts, and, on the other, become familiar with many important results of algebra which are new to him."

The book contains twenty-two chapters. Some notion of what the work contains may be gained from the subject treated under each of the chapters. Chapter I treats of Polynomials and their most Fundamental Properties; Chapter II, a Few Properties of Determinants; Chapter III, The Theory of Linear Dependence; Chapter IV, Linear Equations; Chapter V, Some Theorems concerning the Rank of a Matrix; Chapter VI, Linear Transformations and the Combination of Matrices; Chapter VII, Invariants,—First Principles and Forms; Chapter VIII, Bilinear Forms; Chapter IX, Geometric Introduction to the Study of Quadratic Forms; Chapter X, Quadratic Forms; Chapter XI, Real Quadratic Forms; Chapter XII, The System of a Quadratic Form and One or More Linear Forms; Chapter XIII, Pairs of Quadratic Forms; Chapter XIV, Some Properties of Polynomials in General; Chapter XV, Factors and Common Factors of Polynomials in One Variable and of Binary Forms; Chapter XVI, Factors of Polynomials in Two or More Variables; Chapter XVII, General Theorems on Integral Rational Invariants; Chapter XVIII, Symmetric Polynomials; Chapter XIX, Polynomials Symmetric in Pairs of Variables; Chapter XX, Elementary Divisors and the Equivalence of λ-Matrices; Chapter XXI, The Equivalence and Classification of Pairs of Bilinear Forms and Collineations; Chapter XXII, The Equivalence and Classification of Pairs of Quadratic Forms.

While, in the reading of this book, no algebraic knowledge is presupposed beyond a

familiarity with elementary algebra up to and including quadratics, yet familiarity with some of the higher branches of mathematics, as Analytical Geometry and the Calculus, is essential to understand and master it. The work, as its name implies, is a splendid introduction not only to algebra alone, but to a general course in mathematics. F.

Plane and Spherical Trigonometry. By A. H. Buchanan, LL. D., Professor of Mathematics, Cumberland University. 8vo. Cloth, v+96 pages, 33 figures. Price, $1.00. New York: John Wiley & Sons.

This book aims to comprise about all of trigonometry that is required in most colleges for the A. B. degree. Some of the demonstrations of principles differ slightly from those usually given. The tables are bound in a separate volume. The book, from the printer's stand point, is well gotten up. F.

A Course in Mathematics For Students in Engineering and Applied Science. By Frederick S. Woods and Frederick H. Bailey, Professors of Mathematics in the Massachusetts Institute of Technology. Vol. I. Algebraic Equations, Functions of One Variable, Analytic Geometry, Differential Calculus. 8vo. Cloth, xii+385 pages. Price, $2.00. Boston and Chicago: Ginn & Co.

This book, an entirely new venture, is the first volume of a course in mathematics designed to present in a somewhat more closely connected manner, an amount of mathematical material usually given in distinct courses under the subjects of Algebra, Analytical Geometry, Differential Calculus, and Differential Equations, and covers the work usually required of students in the first two years in colleges and engineering schools. It is believed that such a presentation of the subject will give the student a better grasp of mathematics as a whole. The results of this method of presenting mathematics will be watched with great interest by teachers of mathematics. The book is well gotten up and the material is well arranged and well selected. F.

The Metric and British Systems of Weights, Measures, and Coinage. By F. Mollwo Perkin, Ph. D., Head of Chemistry Department, Borough Polytechnic Institute, London. With Seventeen Diagrams. 8vo. Cloth, 83 pages. Price, 50 cents. London: Whittaker & Co. New York: The Macmillan Co.

This little book will be found of great value in the hands of students studying Chemistry, Physics, Engineering, or General Elementary Science.

In the introduction, the author makes a plea for the use and introduction of the metric system in the British Empire. In several tables of length, area, solid contents, etc., conversion tables are given. Also a chapter on specific gravities is inserted. F.

Integration by Trigonometric and Imaginary Substitution. By Charles O. Gunther, M. E., Assistant Professor of Mathematics and Mechanics in Stevens Institute of Technology, with an Introduction by J. Burkett Webb, C. E., Professor of Mathematics and Mechanics in Stevens Institute of Technology. 8vo. Cloth, vi+79 pages. Price, $1.25. New York: D. Van Nostrand Co.

In this little book the author, by means of what he calls the "Triangle Method," eliminates the *Reduction Formulae*. The student, by the use of this method, becoming practically independent of reduction formulae and tables of integrals. The method is worthy the attention of writers on the Integral Calculus. F.

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EXTERNAL ENCOURAGEMENT FOR THE STUDY OF HIGHER MATHEMATICS.

By DR. G. A. MILLER, University of Illinois.

Among the potent external encouragements for the study of higher mathematics are the international congresses, national societies, prizes, fellowships, and scholarships. In our own country the last two relate mainly to those who are beginners in original mathematical work while the others affect principally those who are continuing such work after the so-called student days. The first formal international mathematical congress was held in Zürich, Switzerland, in 1897, and it was then resolved to hold such congresses about every three years in various countries. Thus far only two others have been held,—one in Paris, France, in 1900, and the other in Heidelberg, Germany, in 1904. The fourth is to be held in Rome, Italy, in April of the present year. The interest in these meetings has been growing rapidly as is partly evidenced by the fact that the published reports contain 306, 450, and 755 pages, respectively.

The oldest mathematical society extant is the one at Hamburg, Germany, having been founded in 1690, and hence it has continued in activity over two hundred years. It can scarcely be called a national organization, especially since the founding of the Deutsche Mathematiker-Vereinigung in 1890, which holds its annual meetings in various parts of the German Empire in connection with the Gesellschaft Deutscher Naturforcher und Arzte. This has become the largest and one of the most influential mathematical societies of the world, even if its fees are only nominal,—2 marks per annum, or 30 marks for life membership. A considerable portion of its 700 members is composed of foreigners so that it, like the other leading societies, is partly international. It publishes a journal, which now appears monthly, and is perhaps best known on account of the extensive reports on the present state of various branches of higher mathematics, which have appeared in it from time to time. Members may receive this journal and occasional other publications at a reduced price.

The London Mathematical Society is the oldest among the prominent

extant societies, having been organized in 1865. It publishes a journal known as the *Proceedings of the London Mathematical Society* which is sent to all the members free of charge. The annual dues are 21 shillings, and there are about 300 members. Next in point of age among the leading societies is the Mathematical Society of France, which was founded in 1872, and had Chasles for its first president. It has published a journal known as the Bulletin since its organization and has always had the leading mathematicians of France among its members even if its growth has been less rapid than that of other prominent societies. Its annual dues are fifteen francs, and its members receive the Bulletin free of extra charge.

Whatever differences of opinion may exist in regard to the relative excellence of the mathematical productivity of various countries, all agree that the countries which use the four languages, French, German, English, and Italian, are pre-eminent and have been pre-eminent for centuries. Hence the student of mathematics who desires to keep well informed in regard to important advances in various lines of mathematical activity must be able to read mathematical works in these four languages. The most active Italian mathematical society of the present is the Circolo Matematico di Palermo which was founded in 1884 and has a membership of 500, among whom are eighty Americans. It is of interest to note that the Americans constitute the largest foreign element in this society, the Germans coming next with fifty-four members. It publishes a journal known as the Rendiconti which is mailed to all its members, the annual dues being the same as those of the French society.

The American Mathematical Society is one of the youngest among the great mathematical societies of the world, having been founded under the name of the New York Mathematical Society in 1888 and hence being only two years older than its German brother. Its growth in numbers has been almost as rapid as that of the German society notwithstanding the fact that its annual dues are ten times as large, being 5 dollars per annum. In view of these larger dues it has been able to render more substantial support to important outside projects, the most noteworthy of these being the publication of the Transactions of the Society in addition to its Bulletin, the latter being sent to all its members free of charge. Both of these journals have maintained a very high standard and have contributed materially towards winning for Americans a favorable recognition among the leading mathematicians of the world. They have also furnished avenues and standards of publication for the younger mathematicians and thus contributed largely to the rapid mathematical development in this country during recent years.

A peculiar feature of the American Mathematical Society is the organization of sections in different parts of our extensive country. Thus far three such sections have been formed — the first at Chicago in 1897, the second at San Francisco in 1902, and the third in the Southwest, starting at Columbia, Mo., in 1906. Each of the first two sections meets twice a year

at or near its place of organization, while the third has had only one meeting a year thus far. These sectional meetings afford opportunities for the members who are located far away from the regular places of meeting of the society, to get together to consider such questions as may tend towards definite advances or to renew courage to proceed in a struggle which must frequently be conducted under adverse circumstances. These meetings and the publications mentioned above are perhaps the most important lines of activity by means of which the American Mathematical Society has centralized and improved the efficiency of mathematical endeavor in our country, but many related minor influences could be mentioned.

Although prizes have not contributed as much to American mathematical progress as to that of Europe, yet the international prizes offered by European institutions have doubtless done considerable to stimulate work of high order in this country, especially since a few of them have come to American mathematicians. The Institute de France is perhaps the best known among the prize bestowing institutions, both on account of its long history and also on account of the eminence of many of its members. Hence scholars appreciate these prizes much more than their money value would imply, being frequently less than five hundred dollars. It is of interest to note that one of the most valued mathematical prizes of the French Institute was awarded to a woman, Sophie de Kowalevski, in the year of the founding of the American Mathematical Society, for the discovery of a new case in which it is possible to integrate the differential equations of the movements of a heavy body having a fixed point.

In most cases these prizes are offered for the solution of a definite problem or for work which tends towards such a solution but in some cases they are given for published work of unusual merit. In 1858 the Grand Prix de Mathématique was offered for the proof of a theorem announced by Legendre in his *Théorie des nombres*, which was afterwards proved to be incorrect, as had been suspected by those who proposed the prize. It may be of interest, in this connection, to mention a very unusual astronomical prize of 100,000 francs which is being offered annually to any one who will establish communication between the Earth and some other star than the planet Mars. It appears that the widow who founded this prize considered communication between Mars and the Earth as too easy to be a subject of this magnificent prize. By a wise provision the interest of this fund may be spent every five years to encourage astronomical research as long as the desired communication is not established.

Fellowships and scholarships have probably constituted the most potent external factor in promoting mathematical research in our country, as they have enabled many young men to get a good start in their scientific careers. It is a sad fact that most of them have accomplished very little beyond the start but the few who continued their work with great energy have more than justified this method of promoting scientific activity.

As fellowships and scholarships are generally reserved for those who hav given evidence of unusual ability and have attained at least the equivaler of the bachelor's degree at a good institution, they render assistanc just when original work can be begun with sufficient promise of success t become fascinating and inspiring.

The first American university to establish graduate fellowships on a extensive scale was Johns Hopkins, which provided for twenty such fellow ships, each yielding $500, at the opening of the institution in 1876. Som of the universities which were opened later, especially Clark and Chicagc adopted similar plans and thus exercised a strong influence to raise th standard of scholarship to a much higher plane. Many of the older institu tions and friends of these institutions have succeeded in establishing equall favorable conditions, which have enabled many young men to get into th research spirit and to learn how to secure new answers from nature, whic is rich but not always liberal. In 1894, G. Stanley Hall published in *Th Forum* a list of graduate scholarships and fellowships at all the America institutions. There were then only eight universities having more than te such positions. Since that time conditions have greatly improved. At th University of Illinois, for example, the amount available for such appoint ments is now more than ten times as large as it was then.

There is a great difference in the demands which the various institutions make upon their graduate scholars or fellows. In some cases the only condition is that the holders of such positions shall reside at the institution and devote all their time to study. In the case of travelling fellowships the incumbent is generally expected to spend his time at one or more of the best institutions in the world along the lines in which he may be especially interested. These scientifically ideal conditions constitute one extreme, the other extreme being presented by the institutions which use these positions to secure a large amount of service, teaching, or reading of papers, at a remarkably low cost and allow the incumbent so little spare time for study as to dwarf him intellectually. The money value of these positions is generally small, varying from free tuition to 700 dollars per year, while a very few specially endowed fellowships pay much more.

For instance, the Kellogg fellowship of Amherst College provides the income of thirty thousand dollars and is awarded for seven years to an alumnus of this college, who shall be selected by the faculty. The first three of the term of seven years shall be spent in study, generally at some foreign institution, and the last four as a lecturer at Amhert College, but the incumbent shall not give more than thirty lectures per annum and shall have no employment except such as pertains to the duty of his fellowship. This fellowship is open to students of mathematics as well as to those interested in other lines, and evidently aims to train the incumbent for a highly scholarly life. It is to be hoped that more positions of this kind will be established, especially in connection with the larger institutions, in order to maintain high ideals among the large student bodies.

A DISCUSSION BY SYNTHETIC METHODS OF THE COVARIANT CONIC OF TWO GIVEN CONICS.*

By PROFESSOR D. N. LEHMER, University of California.

At the close of his chapter on Involution, Professor Reye, in his *Geometry of Position*, gives a proof of the following theorem:

Any two tangents to a conic section which pass through two conjugate points of a given involution of points on a line, intersect, in general, upon another fixed conic.

From the fact that conjugate points in an involution are harmonically separated by the double points of that involution, he is able to state the above theorem in the following way:

The pairs of tangents to a conic section which are harmonically separated by two given points intersect, in general, upon another conic.

This theorem, he notes, is a special case of the following, the proof of which is left to the student:

The pairs of tangents to a conic, which are congugate with respect to a second conic, intersect, in general, upon a third conic.

Just how this last theorem is intended to be developed from the theorems that precede is not clear. It is difficult to make any immediate connection between these last two theorems. Owing to the importance of the last theorem, which concerns, indeed, the covariant conic of the two conics, the following discussion is given.

The locus of poles of the tangents to one conic with respect to a second is a third conic, called the polar reciprocal of the first conic with respect to the second; to four harmonic tangents to the first conic correspond four harmonic points on the polar reciprocal conic. These statements follow from the fact that the tangents to the first conic may be considered as the lines joining corresponding points in two projective point rows. These two point rows reciprocate into two projective pencils of rays, corresponding rays of which meet on the polar reciprocal conic. The second part of the theorem follows easily. We have thus a projective correspondence set up between the tangents of one conic and the points of another. We propose now the following problem, which is fundamental for the purpose in hand:

PROBLEM. *Given a pencil of rays of the second order, and a point row of the second order projectively related to it, to find how many of the lines of the pencil pass through the points of the point row that correspond to them.*

Choose a point S on the front row of the second order as the center of a pencil of the first order perspective to it. This pencil will be projective to the pencil of the second order and the locus of the points of intersection of corresponding rays is a cubic curve with a double point at S. (This

*Read before the meeting of the California Section of the American Mathematical Society February 29, 1908.

is proved in the preceding chapter. See also *The Transactions of the American Mathematical Society*, Vol. 3, pp. 372-376, July, 1902.) This cubic will have at most four points in common with the point row of the second order besides the double point S. These points are easily seen to be the points involved in the problem. We see then, that at most four rays of the pencil pass through the points of the point row that correspond to them.

Equipped with this last theorem we are able to discuss the theorem indicated by Professor Reye: *The locus of points of intersection of the tangents of one conic which are conjugate with respect to another is a third conic.*

Given two conics, α and β. From a point, P, which moves along an arbitrary straight line in the plane draw tangents, PA and PA' to the conic α. We wish to find in how many positions of P on the line the two tangents PA and PA' will be conjugate with respect to β. The two systems of tangents PA and PA' are in involution, so that four harmonic tangents, PA, correspond to four harmonic tangents PA'. The pole of PA, the locus of which is the polar reciprocal of α with respect to β, traces out a point row of the second order projective to PA and thus to PA'. At most four of these poles of PA will therefore lie on PA'. The locus is thus a curve of the fourth degree, being cut by an arbitrary line in at most four points. From the theory of poles and polars, however, if PA' pass through the polar of PA, then will PA' pass through the polar of PA, so that the four points in which an arbitrary line meets the locus coincide in pairs; the quartic is thus a pair of coincident conics.

It is clear that the tangents PB and PB', for a point P on this locus, are harmonic conjugates with respect to PA and PA'. The locus of points from which four harmonic tangents may be drawn to two conics is thus a conic. It is in fact the covariant conic of the two conics. For the analytic side of the discussion, see Salmon's *Conic Sections*, pp. 306 and 344. If the anharmonic ratio of the four tangents be different from -1, the quartic found above does not degenerate necessarily, as appears also from the algebraic discussion. The writer does not know of a discussion by synthetic methods of this remarkable conic, which as the above discussion indicates, passes through the eight points of contact of the four common tangents of the two conics.

JOINT MEETING OF MATHEMATICIANS AND ENGINEERS.

By DR. H. E. SLAUGHT, The University of Chicago.

A series of joint meetings of mathematicians and engineers, conducted at The University of Chicago, December 30, 31, 1907, under the auspices of the Chicago Section of the American Mathematical Society, seemed to inaug-

urate a movement of more than passing importance to both parties concerned. It would seem that these two classes of men have been pursuing paths too widely separated, each class failing to appreciate how large a proportion of their interests may really be counted in the field of the other. Recognition of this community of interests coupled with the great divergence of practice led to a number of significant results: (1) Astonishment that this state of affairs has gone so long with so little serious attention; (2) Gratification over the gathering of such a large and representative body of professors of mathematics, professors of engineering, and practicing engineers for the purpose of thoughtfully considering together the questions of commanding mutual interest; and (3) The determination to take active and far-reaching steps toward harmonizing and strengthening those interests.

The program presented first the historical and statistical phases of the subject both in this country and abroad, and secondly, the pedagogical and practical phases as they appear both to the practicing engineer and to the professor in the engineering school. The discussion was focused about the topic: What is needed in the teaching of mathematics to students of engineering? (a) What range of subjects? (b) To what extent in the various subjects? (c) By what methods of presentation? (d) What should be the chief aim?

The speakers were as follows: Professor E. J. Townsend, department of mathematics, University of Illinois; Professor Alexander Ziwet, department of mathematics, University of Michigan; President R. S. Woodward, Carnegie Institution of Washington; Mr. C. F. Scott, consulting engineer, Westinghouse Electric Company of Pittsburg; Mr. Ralph Modjeski, consulting civil engineer, Chicago, Illinois; Charles S. Slichter, professor of applied mathematics and consulting engineer, University of Wisconsin; Gardner S. Williams, professor of civil, hydraulic, and sanitary engineering, University of Michigan; Frederick S. Woods, professor of mathematics, Massachusetts Institute of Technology, Boston, Massachusetts; George F. Swain, professor of civil engineering, Massachusetts Institute of Technology, Boston, Massachusetts; Arthur N. Talbot, professor of municipal and sanitary engineering in charge of theoretical and applied mechanics, University of Illinois; Fred W. McNair, president, Michigan College of Mines, Houghton, Michigan, and Mr. J. A. L. Waddell, consulting bridge engineer, Kansas City, Missouri.

The general discussion was supported by Mr. C. F. Scott, Pittsburg, Pennsylvania; Dean C. M. Woodward, Washington University; Professor B. F. Groat, School of Mines, University of Minnesota; Professor S. M. Barton, University of the South; President C. S. Howe, Case School of Applied Science; Professor C. A. Waldo, Purdue University; Professor C. B. Williams, Kalamazoo College; Mr. J. B. Webb, consulting engineer, Hoboken, New Jersey; Dean H. T. Eddy, College of Engineering, University of Minnesota; Professor D. F. Campbell, Armour Institute of Technology; Professor A. E. Haynes, College of Engineering, University of Minnesota; Profes-

sor E. W. Davis, University of Nebraska; Professor A. S. Hathaway, Rose Polytechnic Institute; and Professor E. V. Huntington, Harvard University.

Abstracts of the addresses will appear in *The Bulletin of the American Mathematical Society*, and the papers will be printed in full in *Science*.

As an outcome of the discussions, a committee of three was appointed with power to increase the number to fifteen, representing all interests and all sections of the country, who shall make a detailed study of the teaching of mathematics to engineering students in this country, and shall formulate a report to be presented to a joint meeting of mathematicians and engineers to be held in the Summer of 1909 in connection with the annual gathering of the Society for the Promotion of Engineering Education. Those selected for the committee of three were: Professor Gardner S. Williams, University of Michigan, representing the engineering side; Professor E. V. Huntington, Harvard University, representing the mathematical side, and Professor E. J. Townsend, who has already made an extensive study of the data relating to the subject. Among the practical questions already suggested by Professor Townsend for the consideration of all interested are the following:

I. Entrance Requirements.

1. Is a greater uniformity of entrance requirements desirable?
2. Most colleges in the Middle West admit on certificates from accredited schools. In addition to this should engineering students be required to pass an entrance examination in algebra, with the understanding that if they fail to make satisfactory grade, more than the usual amount of work must be done to secure credit in college algebra?
3. Should a knowledge of logarithms and the plotting of simple algebraic curves be added to the entrance requirements?
4. Should the standard of admission be raised so as to include trigonometry and college algebra?
5. Should the requirements be made to cover less ground and intensified?
6. Should more attention be paid to analytic and formal work, particularly in arithmetic and algebra?
7. Should a year of work in mathematics and science of college grade be required for entrance, or should the course be extended to five years?

II. Requirements for Graduation.

1. What should be the relative length of time spent on algebra, trigonometry, analytical geometry, and calculus?
2. Which should precede, algebra or trigonometry?
3. What topics should be particularly emphasized in college algebra? In calculus?
4. How far ought the instruction of the first two years' work in math-

ematics to be made "practical"; how far should we insist upon rigorous demonstrations of principles taught?

5. Should students in one line of engineering, say civil engineering, be given problems of a different nature than those given to students in other lines, say mechanical or electrical engineering?

6. Should differential equations and least squares be required subjects in any engineering course? If so, how extensive should these courses be?

7. Should we have a separate course on "Applications," having for its purpose the cultivation of ability for rapid computation, and the use for engineering work of such instruments as the slide rule, planimeter, integraph, computing machines, etc.?

8. What opportunity for the study of mathematics should be given the engineer beyond the usual course in calculus? What courses might be made elective in the junior or senior years?

9. Should a first course in mechanics be given to engineering students in the freshman year and before the student has had calculus?

III. Administrative Questions.

1. What qualifications should we insist upon for the instructor of engineering students in mathematics?

2. How much elementary mechanics should be taught in connection with the calculus? Should this elementary mechanics be taught by the mathematical department?

3. Should the work in descriptive geometry be made more mathematical in treatment? Should it be taught by the mathematical department?

4. What can be done in general to bring about a closer ralation between the teachers of mathematics and the teachers of engineering?

Pending the report of the Committee of Fifteen, these and other questions relating to the subject may well command the attention of those who wish to promote the scientific and industrial interests of the country as they are related to the training of men who are to be leaders in their development.

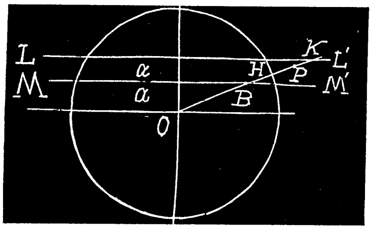
REMARK ON THE TRISECTION PROBLEM.

By E. B. ESCOTT, University of Michigan.

In the article "The Trisection Problem,"* in the May (1907) number of The American Mathematical Monthly; the so-called "Ceroid" is

*The merit of the article referred to lies in the fact that its author inadvertently rediscovered the famous curve and applied it in a new manner. The same author makes use of the hyperbolic curve for trisecting an angle, which of itself is well known, but which is presented in a new form worthy of attention. The Editors.

nothing but the Conchoid of Nicomedes. This can be shown as follows:



In the figure, let LL' be parallel to the x-axis at a distance $2a$, and draw MM' parallel to LL' and half way between it and the x-axis. Let OHK be any line through O, and let P bisect HK. $OB=\frac{1}{2}OK$, $OH=r$. Therefore, $OP=\frac{1}{2}(OH+OK)=\frac{1}{2}r+OB$.

$\therefore OP-OB=BP=\frac{1}{2}r=$constant.

Therefore, the locus of P is a conchoid.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

No satisfactory solutions of Nos. 283, 284, 285, have yet been received. A solution of 283 will appear in the next issue.

287. Proposed by WALTER D. LAMBERT, 416 B Street N. E., Washington, D. C.

For what fraction of a year will there be the greatest difference between the interest as computed by the ordinary commercial rule and that computed by the rule of compound interest?

Solution by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

Let x=the fractional part of a year; r=the rate per annum; \$1=the principal; rx=the simple interest; $(1+r)^x-1$=the compound interest; and y=the greatest difference.

When the time is less than one year the simple interest exceeds the compound interest, therefore

$$y=rx-(1+r)^x+1, \text{ a maximum.}$$

Differentiate, and we have, when m=the modulus,

$$dy/dx=r-(1+r)^x\log(1+r)/m.$$

Equate to zero and we have

$$(1+r)^x\log(1+r)=mr; \text{ or } x=\log[rm/\log(1+r)]/\log(1+r).$$

Also solved by G. B. M. Zerr.

288. Proposed by DR. L. E. DICKSON, Associate Professor of Mathematics, The University of Chicago.

Evaluate the determinant which arises in finding the inverse of the transformation, with binomial coefficients,

$$T:\quad \xi_i=\sum_{j=i}^{g-1}\binom{j}{i}x_j \quad (i=0,\ 1,\ \ldots,\ g-1).$$

Solution by the PROPOSER.

Denote by $D_{n,m}$ the minor of the element in the $(n+1)$th row and $(m+1)$th column. Evidently $D_{nn}=1$, $D_{nm}=0$ for $n<m$. For $n=m+k$, $k>0$,

$$D_{m+k,\,m}=\begin{vmatrix} \binom{m+1}{1} & \binom{m+2}{2} & \binom{m+3}{3}\ldots\ldots\binom{m+k}{k} \\ 1 & \binom{m+2}{1} & \binom{m+3}{2}\ldots\ldots\binom{m+k}{k-1} \\ 0 & 1 & \binom{m+3}{1}\ldots\ldots\binom{m+k}{k-2} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ 0 & 0 & 0\ \ \ldots\ldots\binom{m+k}{1} \end{vmatrix}$$

Studnicka* has evaluated a similar determinant:

$$\begin{vmatrix} \binom{m+1}{1} & \binom{m+1}{2} & \binom{m+1}{3}\ldots\ldots\binom{m+1}{k} \\ 1 & \binom{m+1}{1} & \binom{m+1}{2}\ldots\ldots\binom{m+1}{k-1} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ 0 & 0 & 0\ \ \ldots\ldots\binom{m+1}{1} \end{vmatrix}=\binom{m+k}{k}.$$

Multiplying the first $j-1$ columns of the latter by, respectively,

$$\binom{j-1}{j-1},\ \binom{j-1}{j-2},\ \binom{j-1}{j-3},\ \ldots,\ \binom{j-1}{1},$$

and add the products to the jth column. Then the ith element in the new jth column is† (for $s=j-r$),

$$\sum_{r=i-1}^{j}\binom{m+1}{r-i+1}\binom{j-1}{j-r}=\sum_{s=0}^{j-i+1}\binom{m+1}{j-i+1-s}\binom{j-1}{s}=\binom{m+j}{j-1+1}.$$

Taking $j=k$, $k-1$, …, 2, in turn, we obtain $D_{m+k,\,m}$. Hence

*Cf. Pascal-Leitzmann, *Die Determinanten*, 1900, p. 134.

†Cf. Netto, *Combinatorik*, p. 250, (19); Hagen, *Synopsis*, Vol. 1, p. 65, 5.

$$D_{m+k,\,m}=\binom{m+k}{k},\quad D_{nm}=\binom{n}{m} \text{ if } n>m.$$

Hence the inverse of transformation T is

$$T^{-1}: x_i=\sum_{j=i}^{g-1}(-1)^{i+j}\binom{j}{i}\xi_j \quad (i=0,\ 1,\ \ldots,\ g-1).$$

Since the product of the two transformations is the identity, we have

$$\sum_{j=i}^{l}(-1)^{l+j}\binom{l}{j}\binom{j}{i}=\delta_{il} \quad (\delta_{ii}=1,\ \delta_{il}=0 \text{ if } i\neq l).$$

Conversely, from this well known formula (cf. Netto, p. 255, (43)), follows the evaluation of the determinant D.

289. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that $\left(\frac{n+1}{(n+1)^2-1}+\frac{1}{3}\cdot\frac{n+3}{(n+3)^2-1}+\frac{1}{5}\cdot\frac{n+5}{(n+5)^2-1}+\ldots\right)$ $+\left(\frac{2}{3}\cdot\frac{1}{n+2}+\frac{4}{15}\cdot\frac{1}{n+4}+\frac{6}{35}\cdot\frac{1}{n+6}+\ldots\right)=\frac{n-1}{(n-1)^2-1}+\frac{1}{3}\cdot\frac{n-3}{(n-3)^2-1}$ $+\frac{1}{5}\cdot\frac{n-5}{(n-5)^2-1}+\ldots+\frac{1}{l}\cdot\frac{n-l}{(n-l)^2-1}$, n being any odd integer greater than 1 and $l=n-2$.

Solution by the PROPOSER.

The term by term product of the right hand members of the two Fourier's sine series,

$$1=\frac{4}{\pi}\left[\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\ldots\ldots\right] \text{ and}$$

$$\cos x=\frac{4}{\pi}\left[\tfrac{2}{3}.\sin 2x+\tfrac{4}{15}.\sin 4x+\tfrac{6}{35}.\sin 6x+\ldots\ldots\right]$$

being equal to the product of the left hand members, may be thus written:

$$\cos x=C_0+C_1\cos x+C_2\cos 2x+C_3\cos 3x+\ldots\ldots$$

the C's being constants and functions of the coefficients of the sine terms. But as this product of the right hand members is of the same form as the regular Fourier's cosine series for $\cos x$, it must be identical with the latter series, and, hence, each C except C_1 must be equal to zero. Each C with

even numbered subscript is identically zero, and in order that the remaining C's may be zero, the equation of the problem must hold.

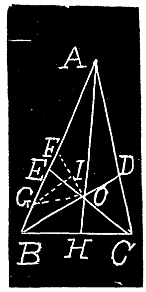
GEOMETRY.

123. Proposed by P. C. CULLEN, Indianola, Iowa.

If the bisectors of two angles of a triangle are equal, those angles are equal, and the triangle is isosceles.

Another Demonstration by JOHN G. GREGG, Terre Haute, Indiana.

Let ABC be the given triangle, and BD, CE, and AH the three bisectors of its angles meeting in O, and let $BD=CE$. We are to show that $\angle ABC=\angle ACB$. If these angles are not equal suppose $\angle ACB>\angle ABC$, then will AB be greater than AC. Take $AG=AC$, and $AF=AD$; then will $OG=OC$ and $OF=OD$, and $\angle GOF=\angle COD=\angle BOE$. Also $OB+OF=BD$...(1), and $OG+OE=CE$...(2).



It can be established that E will always fall between G and F. OB is greater than OG, and OF is greater than, equal to, or less than OG. If $OF>OG$, then also $OF>OE$, and $OB+OF>OG+OE$, or by (1) and (2), $BD>CE$. But by hypothesis, $BD=CE$. Hence $\angle ACB=\angle ABC$. Q. E. D.

Again, if OF is equal to or less than OG, draw GI making $\angle OGI=\angle OBA$. Obviously I will fall between O and F, and the triangles OBE and OGI are similar. Then since $BO>OG$, we have $BO-OE>OG-OI$, and much more, $BO-OE>OG-OF$. Hence $BO+OF>OG+OE$, and from (1) and (2), $BD>CE$ as before, and the theorem is established.

COROLLARY 1. If two lines BD and CE are drawn through a point O in the bisector of an angle, and meeting the sides of the angle, the one (BD) making the less angle with the bisector is the greater.

COROLLARY 2. If the two lines BD and CE make equal angles with the bisector, they are equal.

COROLLARY 3. The line through O, perpendicular to the bisector, is a minimum.

COROLLARY 4. Two triangles are equal if their bases, the angles opposite the bases, and the bisectors of those angles are respectively equal.

320. Proposed by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Prove by plane geometry the following interesting theorem:

If from a point in the plane of a triangle perpendiculars are demitted upon the three sides of the triangle, and if the area of the triangle formed by connecting the feet of these perpendiculars is denoted by $\triangle'$, the distance of the assumed point from the center of the circle circumscribed about the original triangle by R', the radius of the circumscribed circle by R, and the area of the pedal triangle by $\triangle$, then will $\triangle'/\triangle=\pm[(R^2-R'^2)/R^2]$.

I. Solution by the PROPOSER.

There is an error in the statement of this theorem, which, I am afraid, was made by me through an oversight. Instead of the pedal triangle $\triangle$ it should be the triangle whose vertices are the mid-points of the sides of the original triangle. I saw this theorem for the first time, stated but not proved, in a German collection of geometrical exercises, with the following note: "In the English periodical, the *Mathematician,* anno 1844, the editor says that it would be desirable that this remarkable theorem should be established by pure geometry, which would form a good exercise for the student; and that he would be glad to give its investigation, in this way, in some future number of the *Mathematician,* should such be furnished by any of his correspondents." The *Mathematician,* however, has never furnished such a purely geometrical proof. For years I have tried to discover one, but I never succeeded in finding any other proof than the following trigonometrical one.

Let ABC represent the triangle, O the center of its circumcircle, P the assumed point; PL, PM, PN the perpendiculars let fall from it upon the sides of the triangle; OD perpendicular to AC, and OE to AB; PF perpendicular to OD, and PG to OE. Denote the angles of the triangle ABC by A, B, C; $\angle ACP$ by θ, $\angle ABP$ by ϕ, $\angle OPF$ by α, $\angle OPG$ by β.

$\dfrac{MN}{CM}=\dfrac{\sin C}{\cos\theta}$, $\dfrac{LN}{BL}=\dfrac{\sin B}{\cos\phi}$, but $CM=R\sin B+R'\cos\alpha$, and $BL=R\sin C+R'\cos\beta$.

$\therefore MN=\dfrac{\sin C}{\cos\theta}(R\sin B+R'\cos\alpha)$. Let $N=\dfrac{\sin B}{\cos\phi}(R\sin C+R'\cos\beta)$.

$\therefore \triangle'=\frac{1}{2}MN\times LN\sin(\theta+\phi)=\frac{1}{2}.\dfrac{\sin B\sin C}{\cos\theta\cos\phi}\sin(\theta+\phi)\ (R\sin B+R'\cos\alpha)$ $\times\ (R\sin C+R'\cos\beta)=\frac{1}{2}\sin B\sin C(\tan\theta+\tan\phi)\ (R\sin B+R'\cos\alpha)\ (R\sin C+R'\cos\beta)\ldots(1)$.

But in $\triangle OPC$ and $\triangle OPB$, $R':R=\cos(B+\theta):\sin(\theta+\alpha)\ldots(2)$.

$R':R=\cos(C+\phi):\sin(\phi+\beta)\ldots(3)$, and $\alpha+\beta=A\ldots(4)$.

From (2), $\tan\theta=\dfrac{R\cos B-R'\sin\alpha}{R\sin B+R'\cos\alpha}\ldots(5)$, and from (3),

$\tan\phi=\dfrac{R\sin C-R'\sin\beta}{R\sin C+R'\cos\beta}\ldots(6)$.

Substituting these values in (1), we obtain

$$\triangle'=\tfrac{1}{2}\sin B\sin C[R^2\sin A+RR'\cos(B+\beta)+RR'\cos(C+\alpha)-R'^2\sin A].$$

But since $\alpha+\beta=A$, $B+\beta=B+A-\alpha=180^\circ-(C+\alpha)$.

$\triangle'=\frac{1}{2}\sin A\sin B\sin C(R^2-R'^2)$. Since $\sin A=8\triangle/bc$, $\sin B=8\triangle/ac$, $\sin C=8\triangle/ab$, $\sin A\sin B\sin C=512\triangle^3/a^2b^2c^2=2\triangle/R^2$.

$\therefore \triangle'=\dfrac{\triangle}{R^2}(R^2-R'^2)$, or $\dfrac{\triangle'}{\triangle}=\dfrac{R^2-R'^2}{R^2}$. Q. E. D.

Dr. Zerr gave a general discussion of the problem as proposed and also of the problem as intended by the proposer. We give below Professor Schmall's demonstration of the problem as proposed.

II. **Solution by C. N. SCHMALL, A. B., 89 Columbia Street, New York City.**

In Fig. 1, let XYZ be the pedal triangle. Then $AYGZ$ being a cyclic quadrilateral, AG is clearly the diameter of the circle, and we have,

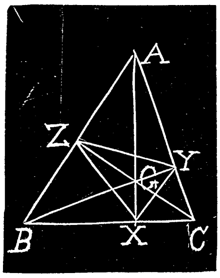


Fig. 1.

$$YZ=AG\sin A=AY\csc AGY\sin A=c\cos A\csc C\sin A$$
$$=a\cos A=R\sin 2A\ldots(1).$$

Similarly, $ZX=b\cos B=R\sin 2B$, and
$XY=c\cos C=R\sin 2C$.
$\therefore$ perimeter $XYZ=\Sigma(a\cos A)=R\,\Sigma\sin 2A$.
Also $\angle YXZ=\angle ABY+\angle ACZ=180^\circ-2A$.
$\therefore$ 2 area $XYZ=XY.XZ.\sin YXZ=bc.\cos B\cos C\sin 2A$.
$\therefore$ area $XYZ=\frac{1}{2}bc\cos B\cos C.2\sin A\cos A=2\triangle''\Pi\cos A$

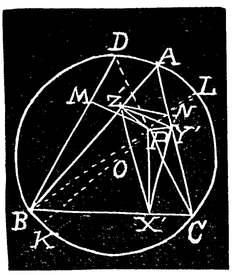
[where $\triangle''$=area of triangle ABC]$=\dfrac{\Pi(a\cos A)}{2R}=\dfrac{\Pi(R\sin 2A)}{2R}$ [by (1) above]

$=\dfrac{R^3\Pi(\sin 2A)}{2R}=\frac{1}{2}R^2\Pi\sin 2A=\triangle$ =area of the *pedal triangle* of triangle ABC.

Now, referring to Fig. 2, let P be the given point (taken within the triangle ABC for convenience).

Let ABC be the given triangle; O the center of its circumscribed circle; X', Y' and Z' the feet of the perpendiculars from P on the sides of the triangle. Let CP (produced) meet the circle in D. Draw BD, and draw PM perpendicular to BD; draw also $Z'N$ perpendicular to $X'Y'$.

Now since $PX'CY'$ is cyclic, and PC is the diameter of its circle, we have



$$X'Y'=PC\sin C\ldots(1),$$

Fig. 2.

(see Phillips and Strong's *Trigonometry*, p. 50; ex. 10); similarly in the quadrilateral $BZ'PX'$,

$$Z'X'=PB\sin B\ldots(2).$$

Also, from the similar right triangles $Z'X'N$ and PBM [which are similar because $\angle Z'X'N=\angle Z'X'P+\angle Y'X'P=\angle Z'BP+\angle Y'CP=\angle Z'BP+\angle DBA=\angle MBP$] we have

$$\frac{Z'N}{PM}=\frac{Z'X'}{PB}=\sin B\ [\text{by }(2)]\ldots(3).\quad \text{Also, } \frac{PM}{PD}=\sin D=\sin A\ldots(4).$$

Multiplying equations (1), (3), and (4) together, we get,

$$Z'N.X'Y'=PC.PD.\sin A\sin B\sin C,\ \textit{i. e.},\ 2\text{ area } X'Y'Z'=PC.PD.\Pi\sin A$$
$$=PL.PK.\Pi\sin A=(OL-OP)(OL+OP).\Pi\sin A$$
$$=(OL^2-OP^2).\Pi\sin A=(R^2-R'^2).\Pi\sin A.$$

If the point P be taken outside the circle of the triangle ABC, then $(OL-OP)$ is negative and thus we get the double sign.

$\therefore \triangle'=\frac{1}{2}(R^2 \pm R'^2).\Pi \sin A$. Hence,

$$\frac{\triangle'}{\triangle}=\frac{\pm\frac{1}{2}(R^2-R'^2).\Pi\sin A}{\frac{1}{2}R^2.\Pi\sin 2A}=\pm\frac{(R^2-R'^2}{R^2}\cdot\frac{\Pi\sin A}{\Pi\sin 2A}.$$

Thus the result, as given by the proposer, is not quite right.

He probably misused the well known formula, $\Sigma \sin 2A=4\,\Pi \sin A$.

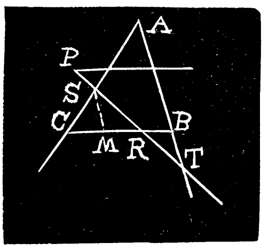
COROLLARY. If the point P lie on the circumscribed circle of triangle ABC, the points X', Y', Z' lie on "Simson's Line" and $\triangle'=0$.

321. Proposed by J. O. MAHONEY, B. E., M. Sc., Central High School, Dallas, Texas.

ABC is an isosceles triangle. Through any point P in its plane draw a line $PSRT$ cutting the sides AC, CB, AB in the points S, R, and T, respectively (R between B and C), so that the segments CS and BT shall be equal.

Solution by C. N. SCHMALL, 89 Columbia Street, New York City.

In the figure, let SM be drawn parallel to AB. Then, if $CS=BT$, $SR=RT$.



Hence to locate the point R we must solve the problem: Given a line drawn through a fixed point, and cutting two fixed intersecting lines, to find the locus of the middle point of the intercepted segment.

Here, we then have $PT-PS=2(PR-PS)$, or $PT+PS=2PR$.

Let (ρ', θ') be the polar co-ordinates of P, and let the equations of AC and AB be

$$a_1x+b_1y+c_1=0\ldots(1),\quad a_2x+b_2y+c_2=0\ldots(2).$$

Transforming to polar co-ordinates these become

$$a_1\rho\cos\theta+b_1\rho\sin\theta+c_1=0\ldots(3),\text{ and } a_2\rho\cos\theta+b_2\rho\sin\theta+c_2=0\ldots(4).$$

From (3), $\rho=-\dfrac{c_1}{a_1\cos\theta+b_1\sin\theta}=PS$; from (4), $\rho=-\dfrac{c_2}{a_2\cos\theta+b_2\sin\theta}=PT$.

Hence the polar equation for the locus of P is, $2\rho=2PR=PT+PS$.

$$\therefore 2\rho=-\frac{c_1}{a_1\cos\theta+b_1\sin\theta}-\frac{c_2}{a_2\cos\theta+b_2\sin\theta}$$

$$-2=\frac{c_1}{a_1\rho\cos\theta+b_1\rho\sin\theta}+\frac{c_2}{a_2\rho\cos\theta+b_2\rho\sin\theta}.$$

Transforming back to rectangular co-ordinates, we have

$$2(a_1x+b_1y)(a_2x+b_2y)+c_1(a_2x+b_2y)+c_2(a_1x+b_1y)=0$$

for the locus of P. This equation represents a hyperbola passing through the vertex A. Hence the intersection of this hyperbola with the base CB will give R, and PR produced will give T.

CALCULUS.

248. Proposed by REV. R. D. CARMICHAEL, Anniston, Ala.

Evaluate $\int_0^{\frac{1}{2}\pi}\sin nx\cot x\,dx$, where n is a positive integer.

II. Solution by FRANCIS RUST, C. E., Pittsburg, Pa.

$$\sin nx=n\cos^{n-1}x\sin x-\binom{n}{3}\cos^{n-3}x\sin^3x+\binom{n}{5}\cos^{n-5}x\sin^5x-\ldots$$

$$\therefore\int_0^{\frac{1}{2}\pi}\sin nx\cot x\,dx=n\int_0^{\frac{1}{2}\pi}\cos^n x\,dx-\binom{n}{3}\int_0^{\frac{1}{2}\pi}\cos^{n-2}x\sin^2x\,dx$$

$$+\binom{n}{5}\int_0^{\frac{1}{2}\pi}\cos^{n-4}x\sin^4x\,dx\ldots\pm\binom{n}{2r+1}\int_0^{\frac{1}{2}\pi}\cos^{n-2r}x\sin^{2r}x\,dx+\ldots$$

Transforming $\int_0^{\frac{1}{2}\pi}\sin^p z\cos^q z\,dz$ by the substitution $\sin z=\sqrt{x}$, we have

$$dz=\frac{dx}{2\sqrt{[x(1-x)]}},\text{ and }\int_0^{\frac{1}{2}\pi}\sin^p z\cos^q z\,dz=\tfrac{1}{2}\int_0^1 x^{\frac{1}{2}(p-1)}(1-x)^{\frac{1}{2}(q-1)}dx$$

$$=\tfrac{1}{2}B[\tfrac{1}{2}(p+1),\tfrac{1}{2}(q+1)].$$

$$\therefore\int_0^{\frac{1}{2}\pi}\sin nx\cot x\,dx,\text{ in beta-functions,}=\frac{n}{2}B\left(\frac{n+1}{2},\frac{1}{2}\right)$$

$$-\tfrac{1}{2}\binom{n}{3}B\left(\frac{n-1}{2},\frac{3}{2}\right)+\tfrac{1}{2}\binom{n}{5}B\left(\frac{n-3}{2},\frac{5}{2}\right)-\ldots$$

Also solved by C. E. White.

250. Proposed by V. M. SPUNAR, East Pittsburg, Pa.

Differentiate $(\log^n x)$.

Solution by C. E. WHITE, Vanderbilt University, Nashville, Tenn.

$\log^n x = \log\log\log\ldots(n \text{ times})\, x.$

$$\therefore d\log^n x = \frac{d(\log^{n-1})}{\log^n x} = \frac{d\log^{n-2}x}{\log^n x \log^{n-1}x} = \ldots = \frac{dx}{\log^n x \log^{n-1}x \log^{n-2}x \ldots \log x . x}.$$

Also solved by J. Scheffer. Some of our readers misinterpreted the meaning of the notation. It should be remembered that the notation means the log of the log of the log, etc., n times, of x. See Byerly's *Integral Calculus*, 2d Ed., p. 2. ED. F.

251. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Find in terms of x the functions c_1x and c_2x defined, respectively, by the relations (a) $x\log(c_1x) = c_1x\log x$,
(b) $x\log x = c_2x\log(c_2x)$.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

(a) We may write (a) thus, $\log(c_1x)^x = \log x^{c_1x}$.

$\therefore (c_1x)^x = x^{c_1x}$, or $c_1x = x^{c_1} = 1 + c_1\log x$

$$+\frac{c_1^2}{2!}(\log x)^2 + \ldots + \frac{c_1^n}{n!}(\log x)^n + \ldots$$

(b) Similarly, we may write (b), $\log x^x = \log(c_2x^{c_2x})$.

$$\therefore x^x = (c_2x)^{c_2x}, \text{ or } c_2x = x^{1/c_2} = 1 + \frac{1}{c_2}\log x + \frac{1}{2!\,c_2^2}(\log x)^2 + \ldots + \frac{1}{n!\,c_2^n}(\log x)^n + \ldots$$

Also solved by J. Scheffer, C. E. White, and V. M. Spunar. Mr. Spuňar, in his solution, used the calculus.

MECHANICS.

131. Proposed by F. P. MATZ.

If the distribution of weight on the foundations of a building is W lb. /(feet)^2, the foundation must be sunk $D = (W/w)\tan 4(\frac{1}{4}\pi - \frac{1}{2}\psi)$ feet deep in earth of density w lb./(feet)^2 and angle of repose ψ.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let E represent the weight of a portion of a horizontal stratum of earth which is displaced by the foundation of a structure, S the utmost weight of that structure consistent with the power of the earth to resist displacement, ψ the angle of repose of the earth. Then $S/E = [(1+\sin\psi)/(1-\sin\psi)]^2$ (see paper "On Stability of Loose Earth," read before the Royal Society on the 19th of June, 1856, and published in the *Philosophical Transactions* for that year). In the problem, $E = Dw$, $S = W$.

$$\therefore \frac{Dw}{W} = \left(\frac{1-\sin\psi}{1+\sin\psi}\right)^2 = \left(\frac{1-\cos(\frac{1}{2}\pi-\psi)}{1+\cos(\frac{1}{2}\pi-\psi)}\right)^2.$$

$$\therefore D=\frac{W}{w}\tan^4(\tfrac{1}{4}\pi-\tfrac{1}{2}\psi),\text{ since }\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=\tan\tfrac{1}{2}\theta.$$

160. Proposed by F. P. MATZ.

Given the para-centric acceleration c^2/r^4 and the angular velocity $(n/m)\pi$ to determine the equation of the orbit.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

The acceleration along the radius vector is given by $d^2r/dt^2-r(d\theta/dt)^2$.

$\therefore d^2r/dt^2-r(d\theta/dt)^2=c^2/r^4$.

The angular velocity is constant and equal to $\pi(n/m)=\beta=d\theta/dt$.

$\therefore d^2r/dt^2-r\beta^2=c^2/r^4$. But $dt=d\theta/\beta$.

$\therefore \beta^2(d^2r/d\theta^2)-r\beta^2=c^2/r^4$. $\beta^2[(dr/d\theta)^2+r^2]-2r^2\beta^2=A-c^2/3r^3$.

$$\therefore \beta^2\left[\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2+\frac{1}{r^2}\right]-\frac{2\beta^2}{r^2}=\frac{A}{r^4}-\frac{c^2}{3r^7}.$$

$$\therefore \frac{\beta^2}{p^2}-\frac{2\beta^2}{r^2}=\frac{A}{r^4}-\frac{c^2}{3r^7},\text{ since }\frac{1}{p^2}=\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2+\frac{1}{r^2}.$$

$\therefore 3\pi^2(n/m)^2r^7+p^2c^2=3p^2[Ar^3+2r^5\pi^2(n/m)^2]$ is the r and p equation.

191. Proposed by DR. L. E. DICKSON, The University of Chicago.

Give the axiomatic principle of Physics which is equivalent to the theorem on the compound of two circles ("Graphical Methods in Trigonometry," MONTHLY, June-July, 1905, pp. 129-133).

Remarks by the PROPOSER.

On page 14 of the present volume, the principle is stated to be that of the parallelogram of forces (or of velocities). This answer is insufficient, as the compound of two circles relates to an infinitude of lines. A complete solution is as follows:

Two forces, represented in magnitude and direction by OP and OR have as their resultant the force represented by the diagonal OQ of the parallelogram $OPQR$. If we take the components of these forces along an arbitrary straight line OS, the sum of the components of OP and OR must equal the component of OR. But in the figure (Vol. XII, top of p. 132), these components are the chords $O\pi$, $O\rho$, OS, respectively. Now $O\pi+O\rho=OS$ yields the point S called for by the definition of the compound of the circles on the diameters OP and OR.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

145. Proposed by J. D. WILLIAMS, being the 12th of his fourteen challenge problems proposed in 1832.

Make $x^2+y^2=\square$, $\frac{5}{4}(x^2+y^2)=$a cube, $xy=2x^3$, $2(x+y)+\frac{xy}{x+y}=\square$, and $(x^4+y^4)(x^2+y^2)-(x^5+y^5)\sqrt{(x^2+y^2)}=\square$.

No solution has been received.

146. Proposed by PROFESSOR JOSE J. CORONADO, Halapa, Vericruz, Mexico.

Find two numbers whose difference is equal to the difference of their cubes.

I. Solution by G. B. M. ZERR, Philadelphia, Pa.; A. H. HOLMES, Brunswick, Me.; and J. E. SANDERS, Reinersville, O.

Let x, y be the numbers. Then $x-y=x^3-y^3$, $1=x^2+xy+y^2$, if $x\neq y$. Let $x=vy$.

$\therefore y=\frac{1}{\sqrt{(v^2+v+1)}}$, $x=\frac{v}{\sqrt{(v^2+v+1)}}$.

Let $v^2+v+1=(nv+1)^2$.

$\therefore v=\frac{1-2n}{n^2-1}$. $\therefore x=\frac{1-2n}{n^2-n+1}$, and $y=\frac{n^2-1}{n^2-n+1}$,

where n can have any value, positive or negative, whole or fractional.

II. Solution by DR. L. E. DICKSON, The University of Chicago.

$x-y=x^3-y^3$. Say $x\neq y$. $\therefore 1=x^2+xy+y^2$.

If any two numbers are desired, there are an infinitude of answers. If two integers are desired (neither zero), then one must be negative, otherwise $x^2+xy+y^2\geqq 3$. Say y is negative, $=-z$. $\therefore 1=x^2-xz+z^2$, x and z positive integers; $\therefore 1-xz=(x-z)^2$; $\therefore 1-xz=0$, or positive.

$xz\leqq 1$. $\therefore x=z=1$. If one is zero, the other$=0$ or ±1.

$\therefore$Only integral sets are $(0, 0)$, $(0, \pm1)$, $(\pm1, 0)$, $(\pm1, \mp1)$.

Combined: (x, y), $\begin{matrix} x=0, & \pm1, \\ y=0, & \pm1. \end{matrix}$

MISCELLANEOUS.

170. Proposed by J. W. NICHOLSON, A. M., LL. D., Baton Rouge, La.

If n and m are any two real numbers whatever, n being less than m, find a rational r such that $\sqrt{n}<r<\sqrt{m}$.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

This depends on how we choose our markings. If we choose the natural numbers 1, 2, 3, 4, etc., $\sqrt{n}$ may be defined by two infinite series of rational numbers, and $\sqrt{m}$ may also be so defined. As these two infinite

series cannot be the same when $n \neq m$, there is some rational number between them.

Let $\sqrt{n}=e+f$, where f is the decimal part, and $\sqrt{m}=g+h$, where h is the decimal part. Then $r=e+k$, where k is any number from 1 to $g-e-1$.

171. **Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.**

If $\lim\limits_{x=a} \dfrac{\phi(x)}{\psi(x)}=\lambda$, show $\lim\limits_{x=a}\left[\dfrac{\lambda}{\phi(x)}-\dfrac{1}{\psi(x)}\right]=\dfrac{\lambda\psi''(a)-\phi''(a)}{2\phi'(a)\psi'(a)}$.

Solution by G. W. GREENWOOD, M. A., Dunbar, Pa.

By hypothesis, $\phi(a)=0$, and $\psi(a)=0$. We must *assume* that $\phi'(a)\neq 0$ and $\psi'(a)\neq 0$. Then

$$\text{Lim.}\frac{\phi(x)}{\psi(x)}=\left[\frac{\phi'(x)}{\psi'(x)}\right]_{x=a}=\lambda. \quad \therefore \lambda\psi'(a)-\phi'(a)=0.$$

$$\text{Lim.}\left[\frac{\lambda}{\phi(x)}-\frac{1}{\psi(x)}\right]=\text{Lim.}\left[\frac{\lambda\psi(x)-\phi(x)}{\phi(x)\psi(x)}\right]=\text{Lim.}\left[\frac{\lambda\psi'(x)-\phi'(x)}{\phi(x)\psi'(x)+\phi'(x)\psi(x)}\right]$$

$$=\text{Lim.}\left[\frac{\lambda\psi''(x)-\phi''(x)}{\phi(x)\psi''(x)+2\phi'(x)\psi'(x)+\phi''(x)\psi(x)}\right]=\frac{\lambda\psi''(a)-\phi''(a)}{2\phi'(a)\psi'(a)}.$$

Also solved similarly by G. B. M. Zerr. Unless one assumes that $\phi(a)=\psi(a)=0$, the problem is not true, as may be easily verified. ED. F.

PROBLEMS FOR SOLUTION.

ALGEBRA.

297. **Proposed by W. J. GREENSTREET, Marling School, Stroud, England.**

If a, b, c, d, f, g, h are all real, and a, $ab-h^2$, $abc+2fgh-af^2-bg^2-ch^2$ are all positive, show that b, c, $bc-f^2$, and $ca-g^2$ are also positive.

GEOMETRY.

330. **Proposed by J. J. QUINN, Ph. D., New Castle, Pa.**

A line pivoted at the origin revolving with a constant angular velocity, intersects another moving parallel to the Y-axis with a constant linear velocity. (1) Find the locus of their intersection when the ratio of their velocities is as $m:n$ referred to a quadrant and a radius, respectively. (2) Assume $m=3$ and $n=2$, and apply to the trisection of an angle. (3) Under what conditions will this curve become a quadratrix? (4) Name the curve.

CALCULUS.

254. Proposed by H. S. PARDEE, Boston, Mass.

A wire is wound in the form of a helix. Assuming that sections of the wire perpendicular to the axis of the wire are circles, find the equation of a section of the wire perpendicular to the axis of the helix.

MECHANICS.

214. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

An inelastic particle is projected in a direction BD from B in a straight line AB. It strikes a rigid line AD in D and returns to AB at C. Find AC/AB, and show on *a priori* ground that this ratio is independent of the velocity of projection.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

152. Proposed by H. S. VANDIVER, Bala, Pa.

When p is a prime of the form $5n\pm1$ then there is a positive integer a such that $a^2\equiv5(\text{mod } p)$. Show that $\left(\frac{a\pm1}{p}\right)=\pm\left(\frac{-2a}{p}\right)$, according as p is of the form $5n+1$ or $5n-1$.

AVERAGE AND PROBABILITY.

195. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue Philadelphia, Pa.

A random diameter is drawn in a given circle. Find the chance that it intersects, (1) a random chord; (2) a random chord through a random point; and (3) a chord through two random points.

NOTES AND NEWS.

We learn from the *Scientific American* of February 1, 1908, that Dr. Paul Wolfskehl, who died recently at Darmstadt, left in his will a provision for the payment of 100,000 marks to the first person who will prove or disprove "the last theorem of Fermat," viz., $x^n+y^n=z^n$, is not possible in integers for $n>2$. F.

The Fourth International Congress of Mathematicians will convene in Rome April 6th to 11th, 1908. Extensive preparations are announced for the entertainment of delegates and their friends. The deliberations of the Congress will be conducted under four sections, each provided with leaders of international reputation: (1) Arithmetic, Algebra, Analysis; (2) Geometry; (3) Mechanics, Mathematical Physics; (4) Philosophical, historical and didactic questions. Professor E. H. Moore, who is now sojourning in Italy, will represent The University of Chicago. S.

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A METHOD OF DERIVING EULER'S EQUATION IN THE CALCULUS OF VARIATIONS.

By DR. GILBERT AMES BLISS, Princeton University.

The derivation of Euler's equation can be made by means of integrals of the form

$$\int [A(x, y)+B(x, y)y']dx, \quad y'=\frac{dy}{dx}, \tag{1}$$

which are independent of the path of integration. Invariant integrals of this type play an important role in the proofs that the usual conditions in the calculus of variations are sufficient to insure a minimum or a maximum, and their introduction in connection with the derivation of Euler's equation makes it possible to simplify considerably the presentation of the whole theory.

In the first section below there is a simple discussion of the conditions under which an integral of the form (1) is independent of the path, and in the second section these results are applied to the derivation of Euler's equation.

§1. Invariant Integrals.

In the integral (1) suppose that the functions $A(x, y)$ and $B(x, y)$ are continuous in a certain region R of the xy-plane. Along an arc C_{12}

$$y=y(x), \quad x_1 \leqq x \leqq x_2, \tag{2}$$

which joins two given points (x_1, y_1) and (x_2, y_2), lies in R, and for which the function $y(x)$ is continuous and has a continuous derivative, the integral I will have a value

$$(3) \qquad I=\int_{x_0}^{x_1}\{A[x,\ y(x)]+B[x,\ y(x)\ y'(x)]\}dx$$

denoted by I_{12} or $I(C_{12})$. If for two arbitrarily chosen points $(x_1,\ y_1)$, $(x_2,\ y_2)$, the values I_{12} are all the same however the arc C_{12} is chosen, the integral I is said to be indepenent of the path.

The sum of the values of such an invariant integral taken along the sides of a triangle of arcs of the type (2) is zero. To prove this, let the vertices of the triangle be denoted by 1, 2, 3, and the sides by C_{23}, C_{31}, C_{12}. It is always possible to pass an arc D of the type (2) through the three points 1, 2, 3, so that for a triangle such as the one in the figure,

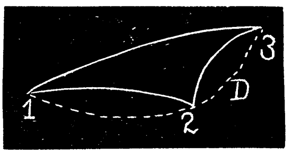

Fig. 1.

$$I(C_{12})+I(C_{23})=I(D_{12})+I(D_{23})$$
$$=I(D_{13})=I(C_{13}).$$

But for any arc C_{13},

$$(4) \qquad I_{13}=-I_{31},$$

since I_{31} is found from an integral of the form (3) by simply changing the limits. Hence along the sides of the triangle

$$I_{12}+I_{23}+I_{31}=0.$$

A similar theorem holds for any polygon of arcs C_{12}, C_{23}, ..., $C_{n-1,1}$, of the type (2). For select a point 0 not on any of the ordinates of the vertices of the polygon, and join it to them by straight lines. Then

$$\begin{aligned} &I_{01}+I_{12}+I_{20}=0,\\ &I_{02}+I_{23}+I_{30}=0,\\ &\quad .\quad\ .\quad\ .\quad\ .\\ &I_{0,\,n-1}+I_{n-1,\,1}+I_{10}=0, \end{aligned}$$

and by adding and using equations similar to (4),

$$I_{01}+I_{12}+\ldots+I_{n-1,\,1}=0.$$

The integral

$$\phi(x,\ y)=\int_{(x_0,\ y_0)}^{(x,\ y)}(A+By')dx$$

taken from the fixed point (x_0, y_0) to the point (x, y) over any continuous curve formed of a finite number of arcs of the type (2) defines a single-valued function $\phi(x, y)$. For any two broken curves joining (x_0, y_0) with (x, y) form a polygon over which the value of the integral is zero, and the two values found by integrating from (x_0, y_0) to (x, y) over the two broken curves, are equal on account of the property (4).

The difference of two values of $\phi(x, y)$ corresponding to points on the same ordinate has the value

$$\phi(x, y) - \phi(x, y_1) = \int_{y_1}^{y} B(x, y)\,dy. \tag{5}$$

This difference can in fact be written in the form

$$\phi(x, y) - \phi(x, y_1) = \int_{(x-h,\, y_1)}^{(x,\, y)} (A + By')\,dx - \int_{x-h}^{x} A(x, y_1)\,dx, \tag{6}$$

since from the figure it is evident that

$$\phi(x, y) = I(C) + \int_{(x-h,\, y_1)}^{(x,\, y)} (A + By')\,dx,$$

$$\phi(x, y_1) = I(C) + \int_{x-h}^{x} A(x, y_1)\,dx.$$

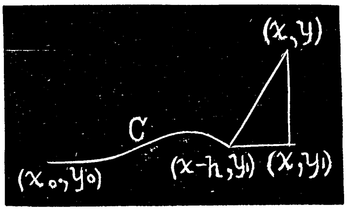

Fig. 2.

Equation (6) may also be written, by a change of variable,

$$\phi(x, y) - \phi(x, y_1) = \int_{(x-h,\, y_1)}^{(x,\, y)} A\,dx + \int_{y_1}^{y} B\,dy - \int_{x-h}^{x} A(x, y_1)\,dx.$$

When h approaches zero the first and third of these integrals vanish, and the second approaches the value given in equation (5).

The derivatives of $\phi(x, y)$ can now be readily calculated. From (5) it follows that

$$\frac{\partial \phi}{\partial y} = B(x, y).$$

It is evident from a figure similar to figure 2 that

$$\phi(x, y) - \phi(x_1, y) = \int_{x_1}^{x} A(x, y)\,dx,$$

so that

$$\frac{\partial \phi}{\partial x}=A(x, y).$$

A necessary and sufficient condition for the integral I to be independent of the path in a region R is therefore that a single-valued function $\phi(x, y)$ *exists in R having the derivatives*

$$\frac{\partial \phi}{\partial x}=A, \quad \frac{\partial \phi}{\partial y}=B. \tag{7}$$

The sufficiency of this condition was not proved above, but follows easily with the help of the fundamental theorem of the integral calculus. For along any arc (2)

$$\int_{(x_1, y_1)}^{(x_2, y_2)}(A+By')\,dx=\int_{(x_1, y_1)}^{(x_2, y_2)}\left(\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}y'\right)dx=\phi(x_2, y_2)-\phi(x_1, y_1).$$

Another criterion for the invariancy of I is the following:

If the functions A and B in the integral (1) *have continuous partial derivatives* $\frac{\partial A}{\partial y}$ *and* $\frac{\partial B}{\partial x}$ *in a simply connected region R*, then a necessary and sufficient condition for the integral I to be independent of the path, is that the equation*

$$\frac{\partial A}{\partial y}=\frac{\partial B}{\partial x} \tag{8}$$

is identically satisfied in R.

The condition is necessary, for from the previous theorem a function ϕ must exist with the derivatives (7), and we have

$$\frac{\partial A}{\partial y}-\frac{\partial B}{\partial x}=\frac{\partial^2 \phi}{\partial x \partial y}-\frac{\partial^2 \phi}{\partial y \partial x}=0.$$

To prove the sufficiency, suppose first that R is a rectangle with one corner at (x_0, y_0). Then the values of I taken from (x_0, y_0) to (x, y) along lines parallel to the x and y axes define a function

$$\phi(x, y)=\int_{x_0}^{x}A(x, y_0)\,dx+\int_{y_0}^{y}B(x, y)\,dy, \tag{9}$$

*R is said to be simply connected if any two of its points can be connected by a continuous curve, and if the interior of any continuous, closed, non-intersecting curve in R is also entirely within the region.

which is readily seen with the help of equation (8) to have the partial derivatives A and B. Consequently in any rectangle where condition (8) is satisfied, the integral I is independent of the path.

In a more general simply-connected region R, the value of I taken around any polygon whose sides are parallel to the x- or y-axis is zero. For by continuing all the sides of the polygon, its interior is divided into rectangles, and the sum of the values of I taken around these rectangles in the positive direction* is the value of I taken in the positive direction about the original polygon. This follows because we have to integrate along a side of any rectangle twice, in opposite directions, unless the side is an edge of the original polygon. But since R is simply connected, the rectangles are all in R when condition (8) holds, and the value of I taken around the edge of such a rectangle is zero. Hence if we integrate from a fixed point (x_0, y_0) to the point (x, y) in R along a broken curve consisting of straight lines parallel to one or the other of the axes, a single-valued function $\phi(x, y)$ is defined. This function like the function defined by equation (9) has A and B for its partial derivatives.

For the application to be made in the next section, the following remark is important.

If the integral (1) *takes the same value over all continuous curves consisting of a finite number of arcs of the type* (2) *joining two fixed points* (x_1, y_1) *and* (x_2, y_2), *then it must be independent of the path in the same way for any two points in the strip of the plane between the ordinates* $x=x_1$ *and* $x=x_2$.

Let 3 and 4 be any two points in this strip, and join 1 with 3, and 4 with 2 by fixed arcs of the type (2). Then however the points 3 and 4 are connected by broken arcs, we will always, by hypothesis, have the same value for

$$I_{13}+I_{34}+I_{42}.$$

Consequently, I_{34} is independent of the path also.

§3. The Derivation of Euler's Equation.

The problem of the calculus of variations which we shall consider here is the problem of finding a curve which joins two given fixed points (x_0, y_0) and (x_1, y_1), and gives a maximum or a minimum value to an integral of the form

$$J=\int f(x, y, y')\,dx.$$

The function f under the integral sign will be supposed to have continuous derivatives of the first and second orders for points (x, y) in a region R of

*$I.\ e.$ keeping the interior of the rectangle on the left.

the xy-plane and all values of y'. Evidently J will then have a well-defined value over any continuous curve consisting of a finite number of arcs of the type (2) in R.

Suppose that one of these arcs

$$(10) \qquad E; \qquad y=y(x), \quad x_0 \leqq x \leqq x_1,$$

joins the points 0 and 1 in R and gives J a minimum value. The family of arcs

$$v; \qquad \bar{y}=y(x)+\alpha\eta(x), \quad x_0 \leqq x \leqq x_1,$$

where α is an arbitrary constant and $\eta(x)$ satisfies the conditions

$$\eta(x_0)=\eta(x_1)=0,$$

also joins the points 0 and 1 and include E for the value $\alpha=0$. If the function $\eta(x)$ has a derivative which is continuous except perhaps at a finite number of points in the interval $x_0 \leqq x \leqq x_1$, then along any arc v the integral J will have a value

$$J(\alpha)=\int_{x_0}^{x_1} f(x, \bar{y}, \bar{y}')\,dx,$$

which is a function of α. Since v reduces to E when $\alpha=0$, it follows that for $\alpha=0$ the function $J(\alpha)$ must have a minimum and $\frac{dJ}{d\alpha}$ must be zero.

The derivative $\frac{dI}{d\alpha}$ for $\alpha=0$ is easily found to be

$$(11) \qquad \left(\frac{dJ}{d\alpha}\right)_{x=0} = \int_{x_0}^{x_1}\left(\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta'\right)dx,$$

where in the derivatives of f the values of y and y' along E are substituted. This derivative must vanish however the function $\eta(x)$ satisfying the conditions given above, is chosen. We see then that in the $x\eta$-plane the integral (11) is independent of the path for all curves joining the points $(x_0, 0)$ and $(x_1, 0)$, and consequently independent of the path anywhere in the strip of the $x\eta$-plane between the ordinates $x=x_0$ and $x=x_1$. There must therefore exist a function $\phi(x, \eta)$ which has the derivatives

$$(12) \qquad \frac{\partial \phi}{\partial x}=\frac{\partial f}{\partial y}\eta, \quad \frac{\partial \phi}{\partial \eta}=\frac{\partial}{\partial y'}.$$

From the first of these equations,

$$\phi = \eta \int_{x_0}^{x} \frac{\partial f}{\partial y} dx + H(\eta), \tag{13}$$

and from the second,

$$\frac{\partial f}{\partial y'} = \int_{x_0}^{x} \frac{\partial f}{\partial y} dx + H', \tag{14}$$

an equation which must be true at every point of E. It is evident by deriving again for η, that H'' must be zero, and H has the form

$$H = c\,\eta + d. \tag{15}$$

If the curve E has corners for the values $x=\alpha, \alpha_2, \ldots, \alpha_n$ in the interval $x_0 \leqq x \leqq x_1$, then the functions (12) may have discontinuities, but they will certainly be continuous in any strip of the $x\,\eta$-plane between two ordinates $x=\alpha_k$, $x=\alpha_{k+1}$. In such a strip ϕ must therefore be continuous and it is evident from equations (13) and (15) that c has the same value throughout the strip.

In two different strips c is also the same. For simplicity consider the case when there is but one value $x=\alpha$, and let

$$\phi = \eta \int_{x_0}^{x} \frac{\partial f}{\partial y} dx + c\,\eta + d, \quad \psi = \eta \int_{x_0}^{x} \frac{\partial f}{\partial y} dx + e\,\eta + f \tag{16}$$

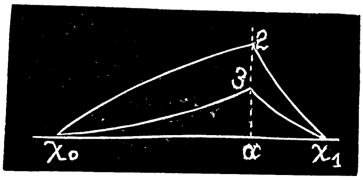

Fig. 3.

be the two functions (13) in the two strips. Along two curves such as those in figure 3,

$$I = \phi_2 - \phi_0 + \psi_1 - \psi_0 = \phi_3 - \phi_0 + \phi_1 - \phi_3,$$

the subscripts indicating the points at which the values of the functions are to be taken. Hence

$$\phi_2 - \phi_3 = \psi_2 - \psi_3,$$

from which it follows with the help of equations (16) that $c=e$.

From equation (14), therefore,

$$\frac{\partial f}{\partial y'} = \int_{x_0}^{x} \frac{\partial f}{\partial y} dx + c, \tag{17}$$

where c has the same value throughout the whole interval $x_0 \leqq x \leqq x_1$.

Several important conclusions can be drawn from equation (17). In the first place consider a point 2 on E when y' is continuous. The left member of (17) may be regarded for the moment as a function of the two varibles y' and x, the latter entering explicitly and also in the function $y(x)$. Then the values x_2, y_2, y'_2 at the point 2 on E furnish a solution of equation (17). If $\frac{\partial^2 f}{\partial y'^2}$ is different from zero for these values, then the theory of implicit functions tells us that equation (17) has but one continuous solution $y'(x)$ which reduces to y'^2 when $x=x_2$, and this solution has a continuous derivative. The values of $y'(x)$ on the curve E near the point 2 constitute this solution, and by differentiating equation (17) for x we derive the following theorem:

If an arc E

$$y=y(x), \quad x_0 \leqq x \leqq x_1,$$

minimizes the integral J, then at any point on E where $y'(x)$ is continuous and $\frac{\partial^2 f}{\partial y'^2}$ different from zero, the function $y(x)$ must also have a second derivative and satisfy the Euler differential equation

$$\frac{d}{dx}\frac{\partial f}{\partial y'}-\frac{\partial f}{\partial y}=0.$$

It is possible to show further from equation (17) that a minimizing arc can not in general have corner points. At a point (x_2, y_2) of E where $\frac{\partial^2 f}{\partial y'^2}$ is different from zero for all values of y', it is evident that the first member of equation (17) can equal the second for at most one value of y', since $\frac{\partial f}{\partial y'}$ is monotomic. Hence it would be impossible for y' to be discontinuous at (x_2, y_2).

An arc E which minimizes the integral J can not have a corner point at any point (x_2, y_2) where $\frac{\partial^2 f}{\partial y'^2}$, is different from zero for all values of y'. If $\frac{\partial^2 f}{\partial y'^2}$ is different from zero at any point in R for all y''s, then no minimizing curve whatsoever with corner points is possible anywhere in R.

THE PRYTZ PLANIMETER.

By DR. A. R. CRATHORNE, University of Illinois.

The object of this article is to call attention to a little known but very interesting and simple mathematical instrument, the Prytz planimeter, or to give its more common name — the hatchet planimeter. It is easily made from a piece of stiff wire which is bent into the form shown in Fig. 1. One end of the wire is ground to a point and the other end is flattened into a chisel edge. The sharp point and this edge should be in the same plane. For a given area a planimeter which is longer than the longest diameter should be used.

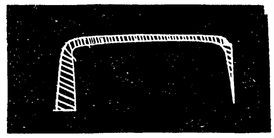

Fig. 1.

To obtain the area of a closed curve with this planimeter a straight line of indefinite length is drawn from the approximate center of gravity in any direction (Fig. 2). The pointed end of the instrument is placed on the center of gravity and the chisel edge on the straight line (at A in the figure). The legs of the planimeter must at all times be perpendicular to the plane of the area. From its position B, the tracing point is now moved along the straight line to the intersection with the boundary curve and then around the area in the direction indicated and back to B. The edged end of the instrument, upon which a slight pressure is brought to bear, traces out the curve $ADEFHIK$ and when the tracing point has returned to B, will take the position K. The product of the length AK into the length of the planimeter will be the approximate area of the given curve (see equation [7]).

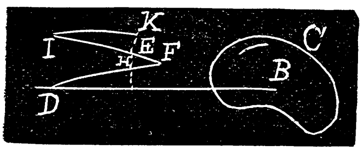

Fig. 2.

This instrument was invented some fifteen years ago by Captain Prytz of the Danish Army, who published an account of his invention in the English magazine, *Engineering*, Vol. 72, page 813. A detailed analytical discussion of its theory was given by M. F. W. Hill in the *Philosophical Magazine* for 1894. In the *Bulletin de l' Académie Imperiale des Sciences de St. Petersbourg*, 1903, Professor Kriloff of the Russian Naval Academy discussed the instrument from the geometrical standpoint and gave a very elementary and simple explanation of its theory.

This theory depends on the well known theorem:* The total area Z swept out by a straight line AB moving in a plane is given by the formula

$$Z = ls + (\tfrac{1}{2}l^2 - al)(\theta_2 - \theta_1), \qquad [1]$$

*See Chapter XIV, Gibson's *Calculus*.

where l is the length of the moving line; s, the total normal displacement of a point P of the line; a, the distance AP; and θ_1, θ_2 are the initial and final values of θ, the angle made by the moving line with some fixed line. If, in particular, the point A moves around a closed curve C', while at the same time B makes a complete circuit of the curve C which lies entirely outside of C', then $\theta_2=\theta_1$, and we have

[2] $$Z=C-C'=ls.$$

The usual conventions as to the signs of the areas in question hold. Areas covered twice in opposite directions by the line are zero, and an area described counter clock wise is positive. If a small wheel with its axis in the moving line be attached at P, the arc through which it turns will give us the normal displacement s. We will call this wheel the measuring wheel. It should be noticed that the position of the point P does not enter into equation [2]. In measuring areas with most planimeters the area of the curve C' is a constant of the instrument. In the case where the point corresponding to A moves backwards and forwards on a curve this constant is zero. In the well known Amsler polar planimeter the curve C' is a circle, and in some other instruments it is a straight line. The area C is the area to be measured.

In the Prytz planimenter there is no fixed curve C', but instead we have a curve which depends upon the curve whose area we are measuring. Referring to figure 2, we see that as the point B moves in its path, the edge A moves in its curve of pursuit from A to K. Now turn the planimeter horizontally about the point B until it is in the initial position AB. The curve corresponding to the curve C' is now a closed curve $ADEFHIKA$. The total area swept out by the line AB is equal to the algebraic sum of the area C and the areas ADE, EFH, and HIK. The other parts of the plane swept by the line are covered twice in opposite directions and hence do not enter into the algebraic sum. Putting in the proper signs we have

[3] $$\text{Total area swept out}=C-ADE+EFH-HIK.$$

From equation [2] the total area swept out is measured by the product of the length l of AB into the length of arc s through which a measuring wheel at A on the line AB would have turned. During the motion of the tracing point this wheel does not turn at all, for the direction of motion is perpendicular to the edge of the wheel. But in turning the instrument about B from the position KB into AB, this wheel will turn through an arc which is equal in length to $l\phi$ where ϕ is the angle KBA. Or we have

[4] $$s=l\phi.$$

The total area swept out by AB is ls or $l^2\phi$. This gives the exact equation,

[5] $$l^2\phi = C - ADE + EFH - HIK.$$

If the starting point for tracing the figure be taken as above (*i. e.* at the center of gravity of the area), the algebraic sum of the three areas enclosed by the curve of pursuit and the arc KA will be very nearly zero and we have the approximate equation

[6] $$c = l^2\phi = l.\text{arc}\,KA.$$

If the angle ϕ be small, say less than 20°, the arc KA can be replaced by its chord, and we have

[7] $$c = l.KA.$$

If an area whose longest diameter is four inches or less, be measured with a ten inch planimeter, the error is very small and is about equal to the error made in finding the area of an equivalent rectangle by measuring the sides with a scale. The error due to the non-alignment of the edge and tracing point can be eliminated by tracing the curve in opposite directions and finding the mean of the two results.

An improved planimeter of this type has a small chisel-edged wheel instead of the chisel edge.

EXISTENCE OF A MINIMUM OF A QUADRATIC FUNCTION.

By T. H. HILDEBRANDT, The University of Chicago.

Suppose we have a quadratic function in n variables,

$$F[x_1, \ldots, x_n] = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, x_i\, x_j + \sum_{i=1}^{n} b_i\, x_i + c,$$

of which we know that

$$F[x_1, \ldots, x_n] \geqq 0,$$

for all values of the variables $x_1, \ldots, x_n$. For instance, such a function is

$$\int_0^{2\pi} [\phi(x) - \sum_{i=1}^{n} (a_i \cos ix + b_i \sin ix)]^2 dx,$$

where the a_i and b_i take the place of x_i. It might appear that because the function has a lower bound, it also has a minimum, *i. e.*, reaches its greatest lower bound. But this statement is not sufficient to establish the fact that it reaches its greatest lower bound in a finite point. If the x_i were restricted to be finite, we could apply the theorem on continuous functions of n variables, which states that in such a case the function has a minimum. But the theorem does not apply when the x_i are permitted to assume infinite values also. We proceed to show that even in this case there is at least one point in the finite part of n-space at which F takes on its minimum value.

Transform F by a linear homogeneous substitution:

$$x_i = \sum_{j=1}^{n} a_{ij}\, x'_i\, x'_j \qquad i=1, \ldots, n,$$

of such a nature that the determinant $|a_{ij}|$ of the substitution is different from zero and so that in the transformed function f the cross product terms $x'_i\, x'_j$ disappear. This is always possible, as a matter of fact, even in the special case where the substitution is orthogonal and therefore $|a_{ij}|=1$. Our function then takes the form:

$$\sum_{i=1}^{n} a'_i\, x'^2_i + \sum_{i=1}^{n} b'_i\, x'_i + c = f(x'_1, \ldots, x'_n),$$

while $f(x'_1, \ldots, x'_n) = F(x_1, \ldots, x_n) \geqq 0$. Then we have $a'_i \geqq 0$ $(i=1, \ldots, n)$. For, suppose $a_i' < 0$. Then let $x'_1, \ldots, x'_{j-1}\, x'_{j+1}, \ldots, x'_n$ take the zero values. We have

$$f(0, \ldots, 0, x_j, 0, \ldots, 0) = a_j'\, x_j^2 + b_j\, x_j + c.$$

But evidently since $a'_j < 0$ we may choose x'_j so large that this becomes negative, contrary to the hypothesis that f shall be greater than or equal to zero for all values of the x'_j. Moreover, if $a'_j=0$, then $b'_j=0$. Suppose $a'_j=0$ and $b'_j<0$. Then by taking for $x'_1. ,.., x'_{j-1}, x'_{j+1}, \ldots, x'_n$, the values zero, we can find a sufficiently large positive value for x'_j to make $b'_j\, x'_j + c'$ negative, and hence, f negative. Hence, if $a'_j=0$, then $b'_j=0$. Our function is then of the form:

$$f(x'_1, \ldots, x_r') \equiv \sum_{i-1}^{r} [a'_i\, x'^2_i + b'_i\, x'_i + c] \qquad (r \leqq n).$$

Transform this function by the substitution

$$x_i'' = x'_i + \frac{b_i}{2a_i}.$$

Then we obtain

$$F(x_1, \ldots, x_n) \equiv \phi(x_1'', \ldots, x_n'') \equiv \sum_{i=1}^{r} a_i' \, x_i''^2 + c' \geqq 0.$$

Evidently $c' \geqq 0$. Moreover we now see at once the minimum value of ϕ, namely,

$$x_1''=0, \ldots, x_n''=0.$$

By putting $x_i''=0$ and solving our transformations backward we can obtain the values of $x_1, \ldots, x_n$ which will make $F(x_1, \ldots, x_n)$ a minimum. There will be a unique solution if $r=n$. For, then we will have n equations in n unknowns. If $r<n$, the solution is not unique, but there will be at least a single infinity of values which will make F a minimum. When $r=n$, our function f evidently has the form:

$$f(x'_1, \ldots, x'_n) = \sum_{i=1}^{n} (a'_i \, x'^2_i + b'_i \, x'_i + c).$$

Now $\sum_{i=1}^{n} a'_i \, x'^2_i$ is a positive, definite form, and so $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \, x_i \, x_j$ must also be. That is, the condition that

$$F(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \, x_i \, x_j + \sum_{i=1}^{n} b_i \, x_i + c \geqq 0$$

may have a minimum is that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \, x_i \, x_j$$

be a positive definite form. This is exactly the result obtained by applying the methods of the differential calculus.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

253. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Prove that $x^5+ax+b=0$ is solvable by radicals if $b=ma$, m being the negative of half the sum of any two roots of the original equation. Exhibit the solution.

Solution by DR. L. E. DICKSON, The University of Chicago.

Let r and s be the two roots for which $m=-\frac{1}{2}(r+s)$. Then

$$r^5+ar+ma=0,\ s^5+as+ma=0.$$

Adding, we get $r^5+s^5=0$. We exclude the trivial case $m=0$, so that $r+s\neq 0$, $r=-es$, e being an imaginary fifth root of unity. From $r+es=0$, $r+s=-2m$, we get

(1) $$s=\frac{2m}{e-1},\ r=\frac{-2me}{e-1}.$$

Now s will indeed be a root if and only if

$$32m^4+al=0,\ l=(e+1)(e-1)^4.$$

Since $e^4+e^3+e^2+e+1=0$, we get $l=5(e^3+e^2+1)$. Now e^2+e^3 and $e+e^4$ are the roots $\frac{1}{2}(-1\pm\sqrt{5})$ of $z^2+z-1=0$. Hence the problem stated is possible if and only if

$$64m^4+5(1\pm\sqrt{5})a=0,\ a=\tfrac{16}{5}(1\mp\sqrt{5})m^4.$$

For m arbitrary, and for this value of a, the equation $x^5+ax+ma=0$ has the two roots (1); the remaining roots may be found by solving a cubic.

284. Proposed by DR. E. H. MOORE, The University of Chicago, Chicago, Ill.

Discuss the system of equations:

$$\begin{cases} x^k+y^k=a_k \\ x^l+y^l=a_l \end{cases} \quad (k,\ l \text{ distinct positive integers})$$

in general and for particular values of $(k,\ l;\ a_k,\ a_l)$.

Solution by BENJAMIN F. FINKEL, Ph. D., Drury College, Springfield, Mo.

§1. GENERAL DISCUSSION.

Assume $k<l$. Solve the first equation for y, say, and we have $y=(a_k-x^k)^{1/k}$. Substitute this value of y in the second equation and free the resulting equation from fractional exponents. The equation thus obtained is called the *Resultant Equation** in x, and is, at most, of the klth degree. It has, therefore, kl roots, real, imaginary, or infinite.

If k and l are both odd or both even, the resultant equation is of the $kl-k$th degree and there are kl roots, k of which are infinite, and, therefore, $kl-k$ roots, real or imaginary.

If k is odd and l even, or if k is even and l odd, the resultant equation is of the klth degree and there are kl roots.

Since the equations are symmetric in x and y the resultant equation for y is the same as that for x. Hence x and y have the same series of values, but not every value of y can be taken with every value of x and satisfy the given system of equations. The final test of the legitimacy of every value of x and y is that these values when substituted in the given system of equations shall satisfy them, and every such set of values of x and y constitutes a solution.

Since the two equations of the system are symmetric with respect to x and y, it follows that if $(x, y)\equiv(a, b)$, $a\neq b$, is a solution, then $(x, y)\equiv(b, a)$ is also a solution. Since, as we have seen above, the greatest number of sets of values of x and y satisfying the given equations is kl, it follows that these sets are made up of sets of values of x and y in which x and y have different values, the number of such sets not exceeding $\frac{1}{2}kl$, and sets of values of x and y in which x and y have the same value.

Let us assume that there are kl solutions of the given system of equations and let $2n'$ be the number of sets of values of x and y wherein x and y have different values, *e. g.*, $(x,y)\equiv(a, b)$, $a\neq b$, and let n'' be the number of sets wherein x and y have the same values, *e. g.*, $(x, y)\equiv(c, c)$.

Then we must have

$$2n'+n''=kl.$$

kl is of the form $2m$ or $2m+1$.

(1) Assume kl is of the form $2m$. Then we have $2n'+n''=2m$, the solutions of which are

n'	n''
m	0
$m-1$	2
$m-2$	4
$\vdots$	$\vdots$
0	$2m$

*Chrystal's *Algebra*, Part I, p. 403, ed. 1885.

(2) Assume kl of the form $2m+1$. Then we have $2n'+n''=2m+1$, the solutions of which are

n'	n''
m	1
$m-1$	3
$m-2$	5
⋮	⋮
0	$2m+1$

Hence, it follows that when kl is even the number of solutions in which x and y are equal must be even, and when kl is odd, the number of such solutions must be odd.

The graph of the equation $x^n+y^n=a_n$, when n is even and greater than 2 and $a_n>0$ is of the form of Fig. 1. When $n=2$ and $a_n>0$, the graph is a circle.

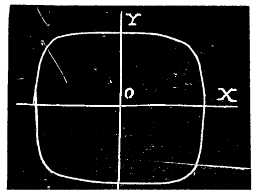

Fig. 1.

When n approaches an infinitely large even number and a_n is a finite positive number, the graph approaches a square whose sides are 2. When $a_n<0$, the graph is imaginary.

The graph of the equation $x^n+y^n=a_n$, when n is odd and greater than 1 is of the form of Fig. 2. When n is 1, the graph is a straight line. When n approaches an infinitely large positive odd integer and a_n is a finite positive number, the graph approaches the full-line curve in Fig. 3. When n approaches an infinitely large odd integer and $a_n<0$, the graph approaches the dotted line in Fig. 3.

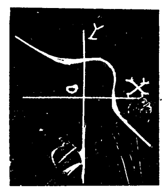

Fig. 2.

If, instead of solving one of the equations with respect to one of the unknowns and substituting this value in the second, to obtain the resultant equation in the other unknown, one solves the two equations for the ratio, r, of x to y, there results, in general, an equation of the klth degree in r. And since the two equations are symmetric in x and y, this equation in r will be a *reciprocal equation.* This fact often enables one to obtain solutions more easily than by the first method. It also enables one to pair the proper values of x and y. It does not, however, prevent the introduction of extraneous roots. Thus, after r is found, and one wishes to find x, say, there will be for each of the kl values of r as many values of x as there are units in the exponent of x in the equation in which r is substituted. Thus, suppose $x^k(1+r^k)=a_k$; from this equation one would obtain k values of x for each of the kl values of r. One would thus obtain k^2l values in all for x, of which not more than kl can be legitimate values, that is, values which when used with certain values of y will satisfy the system of equations.

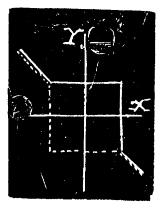

Fig. 3.

§2. Special Cases.

a. If $k=l$ and $a_k=a_l$, then there are an infinitude of solutions. Geometrically interpreted, the two equations represent two coincident curves. Hence, the co-ordinates of all points of the curves will satisfy the two equations.

If $k=l$ and $a_k \neq a_l$, the roots are all infinite (real or imaginary). Thus, let $k=l=1$ and, therefore, $x+y=a$ and $x+y=b$, $a \neq b$. These equations are frequently called inconsistent, but for the sake of uniformity of language they are commonly said to be satisfied by the point at infinity. $(x, y) \equiv (\infty, -\infty)$, $(-\infty, \infty)$ are solutions. It thus appears that there are two solutions here. We may, however, consider this to be really only one solution; for geometrical considerations have led us to regard $-\infty$ and ∞ as the same point if the real number system be represented by points in a straight line.* The equations represent two parallel straight lines.

Suppose $x^2+y^2=a^2$ and $x^2+y^2=b^2$, $a \neq b$. Here, $x=\pm\infty$ and hence $y=\pm\infty i$. Since x and y are symmetric we have 2^3 solutions. These are, $(\infty, \infty i)$, $(\infty, -\infty i)$, $(-\infty, \infty i)$, $(-\infty, -\infty i)$, $(\infty i, \infty)$, $(\infty i, -\infty)$, $(-\infty i, \infty)$, and $(-\infty i, -\infty)$. But from the previous considerations, these eight solutions may be reduced to four, the requisite number; for we may consider $(\infty, \infty i)$, $(\infty, -\infty i)$ as one point, and $(-\infty, \infty i)$, $(-\infty, -\infty i)$ as a second point and so on. If we assume that parallel lines in space intersect at infinity, these four solutions may be reduced to two solutions, the first four above constituting one solution and the last four a second solution.

By a more generally accepted convention, we may consider the two solutious as the "circular points at infinity," through which two points every circle passes. The co-ordinates of one of the circular points at infinity may be considered to be $(\infty, \infty i)$ or $(-\infty, -\infty i)$; these co-ordinates, by the previous consideration, being one and the same point, and the co-ordinates of the other point at infinity being $(\infty, -\infty i)$ or $(-\infty, \infty i)$. These points may be constructed geometrically by the method suggested in Carr's *Synopsis of Pure and Applied Mathematics*, pp. 674-677.

Suppose $x^3+y^3=a$, $x^3+y^3=b$, $a \neq b$. Here, $x=\infty$, $\omega\infty$, $\omega^2\infty$, and $y=-\infty$, $-\omega\infty$, $-\omega^2\infty$. There are here eighteen algebraic solutions, but these, as before, may be reduced to nine, the requisite number.

Suppose $x^4+y^4=a$, $x^4+y^4=b$, $a \neq b$. Here, $x=\pm\infty$, $\pm\infty i$, and $y=\pm\infty\sqrt{i}$, $\pm\infty i^{\frac{3}{2}}$. There are here sixteen solutions, the requisite number.

b. Suppose $k=1$, $l=2$, and, therefore, $kl=2$. Here, $2n'+n''=2$. There are two possibilities:

n'	n''
1	0
0	2

*Reye, *Geometrie der Lage*, p. 18.

As an example of the first possibility, let the system be

$$\begin{cases} x+y=3, \\ x^2+y^2=5, \end{cases} \quad (x,\ y) \equiv (1,\ 2),\ (2,\ 1).$$

In order that the second possibility be satisfied, a_1^2 must equal $2a_2$. As an example of the second possibility, let the system be

$$\begin{cases} x+y=6, \\ x^2+y^2=18, \end{cases} \quad (x,\ y) \equiv (3,\ 3),\ (3,\ 3).$$

c. $k=1$, $l=3$, and, therefore, $kl=3$. Here, $2n'+n''=3$. There are two possibilities:

n'	n''
1	1
0	3

If we solve the system, $\begin{cases} x+y=a_1, \\ x^3+y^3=a_3, \end{cases}$ we find $(x,\ y) \equiv (\infty,\ \infty)$,

$$\left(\frac{a_1}{2} \pm \tfrac{1}{2}\sqrt{\frac{4a_3-a_1^3}{3a_1}},\ \frac{a_1}{2} \mp \tfrac{1}{2}\sqrt{\frac{4a_3-a_1^3}{3a_1}}\right).$$

(1) If $a_1=3$, $a_3=9$; $(x,\ y) \equiv (\infty,\ \infty)$, $(1,\ 2)$, $(2,\ 1)$, an example of the first possibility. The graph of this system is Fig. 4.

(2) In order that the second possibility be satisfied, a_1^3 must equal $4a_3$ or $a_1=0$.

Let $a_1=6$, and $a_3=54$. Then $(x,\ y) \equiv (\infty,\ \infty)$, $(3,\ 3)$, $(3,\ 3)$. The graph of the system is Fig. 5.

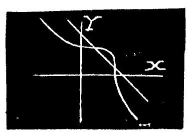

Fig. 4.

d. $k=1$, $l=4$, and, therefore, $kl=4$. Here, $2n'+n''=4$. There are three possibilities:

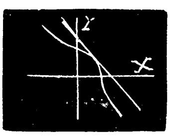

Fig. 5

n'	n''
2	0
1	2
0	4

Solving the system of equations $x+y=a_1$, $x^4+y^4=a_4$, we find (x, y)

$$\equiv \left(\frac{a_1 \pm \sqrt{[-3a_1^2 \mp 2\sqrt{(2a_1^4+2a_4)}]}}{2},\ \frac{a_1 \pm \sqrt{[-3a_1^2 \pm 2\sqrt{(2a_1^4+2a_4)}]}}{2}\right).$$

(1) If $a_1=3$, $a_4=17$; $(x, y)\equiv(1, 2)$, $(2, 1)$, $(3\pm\sqrt{-55}, 3\mp\sqrt{-55})$, an example of the first possibility. The graph of the system is the Fig. 6.

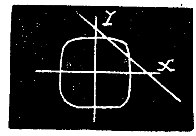

Fig. 6.

(2) For the second possibility, a_1^4 must equal $8a_4$.

If $a_1=4$ and $a_4=32$, $(x, y)\equiv(2,2)$, $(2,2)$, $(2\pm2\sqrt{-6}, 2\mp2\sqrt{-6})$.

a_1 and a_4 must be so taken that $2a_1^4+2a_4$ is a perfect square. Thus, if $a_1=1$ and $a_4=8$, we have an example of the first possibility. If $a_1^4=-a_4$, we have an example also satisfying the first possibility, but since the graph of $x^4+y^4=-1$ is imaginary, all the roots are imaginary.

(3) The only example satisfying the third possibility is $x+y=0$, and $x^4+y^4=0$. Here, $(x, y)\equiv(0, 0)$, $(0, 0)$, $(0, 0)$, $(0, 0)$.

e. $k=1$, $l=5$, and, therefore, $kl=5$. Here, $2n'+n''=5$. There are three possibilities:

n'	n''
2	1
1	3
0	5

Solving the system of equations, $x+y=a_1$, $x^5+y^5=a_5$, we find $(x, y)\equiv(\infty, \infty)$,

$$\left[\tfrac{1}{2}\left(a_1 \pm\sqrt{-a_1^2\mp\tfrac{2}{5}\sqrt{[(5a_1^5+20a_5)/a_1)}}\right),\right.$$

$$\left.\tfrac{1}{2}\left((a_1\pm\sqrt{-5a_1^2\mp\tfrac{2}{5}\sqrt{[(5a_1^5+20a_5)/a_1]}}\right)\right].$$

(1) If $a=3$ and $a_5=33$, $(x, y)\equiv(1, 2)$, $(2, 1)$, $[\tfrac{1}{2}(3\pm\sqrt{-19}), \tfrac{1}{2}(3\mp\sqrt{-19})]$, (∞, ∞).

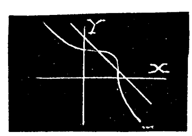

Fig. 7.

(2) For the second possibility, a_1^5 must equal $16a_5$, and $(5a_1^5+20a_5)/a_1$ must be a perfect square. If $a_1=4$ and $a_5=64$, $(x, y)\equiv(2, 2)$, $(2, 2)$, $(2\pm2\sqrt{-2}, 2\mp2\sqrt{-2})$, (∞, ∞).

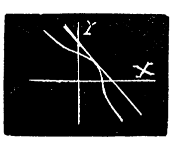

Fig. 8.

(3) The only example satisfying the third possibility is $x+y=0$, $x^5+x^5=0$. Here, $(x, y)\equiv(0, 0)$, $(0, 0)$, $(0, 0)$, $(0, 0)$, (∞, ∞).

f. $k=1$, $l=6$, and, therefore, $kl=6$. Here, $2n'+n''=6$. There are four possibilities:

n'	n''
3	0
2	2
1	4
0	6

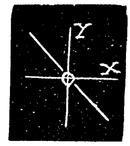

Fig. 9.

The solution of the system $x+y=a_1$, $x^6+y^6=a_6$ leads to a cubic for xy, As the values of x and y are quite complicated, we shall omit them in this case.

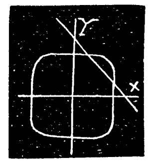

Fig. 10.

(1) If $a_1=3$ and $a_6=65$, $(x, y)\equiv(1, 2)$, $(2, 1)$, (α, β), (β, α), (γ, δ), (δ, γ), the last four sets being imaginary.

(2) If $a_1=2$ and $a_6=2$, $(x, y)\equiv(1, 1)$, $(1, 1)$, (α, β), (β, α), (γ, δ), (δ, γ).

(3) The only equation satisfying the third possibility is $x+y=0$, $x^6+y^6=0$.

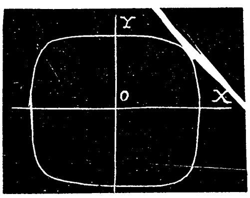

Fig. 11.

(4) The only equations satisfying the fourth possibility are $x+y=0$, $x^6+y^9=0$. There are six solutions each equal to $(x, y)\equiv(0, 0)$.

g. $k=2$, $l=3$, and, therefore, $kl=6$. Here, $2n'+n''=6$.

There are four possibilieies:

n'	n''
3	0
2	2
1	4
0	6

To solve the system $x^2+y^2=a_2$, $x^3+y^3=a_3$, requires the solution of a cubic in xy. If we let $y=rx$, we get the equation,

$$(a_2^3-a_3^2)(r^3+\frac{1}{r^3})-3a_3^2(r+\frac{1}{r})+2a_2^3=0, \text{ or } (a_2^3-a_3^2)u^3-3a_2^3u+2a_2^3=0,$$

where $u=(r+\dfrac{1}{r})$.

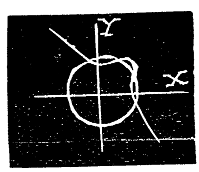

Fig. 12.

(1) If $a_2=13$ and $a_3=35$, $(x, y)\equiv(2, 3)$, $(3, 2)$, $[\frac{1}{2}(2\pm\sqrt{22}), \frac{1}{2}(2\mp\sqrt{22})]$, $[\frac{1}{2}(-7\pm i\sqrt{23}), \frac{1}{2}(-7\mp i\sqrt{23})]$.

(2) In order that the second possibility be satisfied, a_2^3 must equal $2a_3^2$.

If $a_2=8$ and $a_3=16$, $(x, y)\equiv(2, 2)$, $(2, 2)$, $\{\pm[\sqrt{(4\pm2\sqrt{3})}+\sqrt{(\mp2\sqrt{3})}], \pm[\sqrt{(4\pm2\sqrt{3})}-\sqrt{(\mp2\sqrt{3})}]\}$.

(3, 4) In order that the third possibility be satisfied, a_2 and a_3 must both be equal to 0, and these are the only values of a_2 and a_3 which will also satisfy the fourth possibility.

Also solved by G. B. M. Zerr.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

120. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Find the prime numbers p for which $x^2-pxz-px-z+p^2-3=0$ has more than two sets of positive integral solutions x, z, each $<p$.

Solution by H. S. VANDIVER, Bala, Pa.

About the time this problem was originally proposed, Professor Dickson wrote to me that he encountered it in determining those factors of $(p^2-1)^2$ which are of the form $1+px$, p prime. It will first be shown that the latter problem is equivalent to finding all the integers p, x, and z (p a prime) which satisfy

$$x^2-pxz-px-z+p^2-3=0 \ldots (1).$$

Let $(p^2-1)^2=(1+px)(1+py)$, where x and y are positive integers. Expanding and dividing by p,

$$p^3-2p=x+y+pxy.$$

Hence there is necessarily an integer $z>0$, such that $x+y=pz$. By substitution and division by p,

$$z(px+1)=x^2+p^2-2.$$

Setting $z+1$ for z we obtain (1), and the two problems are equivalent.

Solving (1) for p,

$$2p=x(z+1)\pm\sqrt{[x^2(z+3)(z-1)+4(z+3)]}.$$

For this to hold, there is necessarily a positive integer α such that

$$x^2(z+3)(z-1)+4(z+3)=\alpha^2,$$

subject to the condition that $x(z+1)\pm\alpha$ be double a prime.

Putting $z+3=v$, $\alpha^2-vx^2(v-4)=4v$.

Let $v=k^2u$, where u is an integer not divisible by a square other than unity, then

$$u\beta^2-x^2(k^2u-4)=4\ldots. (2),$$

where $\beta=\alpha/ku$, necessarily an integer.

The preceding analysis shows that (1) is soluble only for cases in which

(2) is soluble. It will suffice then to indicate a method leading to the complete solution of (2).

Regard u and k in (2) as fixed, and consider the relation as the representation of the integer 4 by a quadratic form in β and x, with coefficients u and k^2u-4. Consider the case where the determinant is of the form $k^2u^2-4u>4^2$. Then all the solutions of (2) can be obtained by using the theorem given in Serret's *Cours D'Algebre Supêrieure* (Vol. I, p. 80). Suppose, first, that u and k are both odd; then the continued fraction to be considered is:

$$\sqrt{\frac{k^2u-4}{u}}=[k-1;\ 1,\ u'-1,\ 2,\ v'-1,\ 1,\ 2(ku-1),\ 1,\ v'-1,\ 2,\ u'-1,\ 1,\ 2(k-1);\ 1, \ldots],$$

where $ku=2u'+1$, $k=2v'+1$, $k>4$. (As a check, note that the period is reversible, $2(ku-1)$ being the middle term). If u is divisible by 2 but not by 4, and if k is odd, (2) reduces to the form

$$t\beta^2-x^2(k^2t-2)=2, \text{ where } u=2t,\ t \text{ not congruent to } 0 \pmod 2,$$

and the development is

$$\sqrt{\frac{k^2t-2}{t}}=[k-1;\ 1,\ kt-2,\ 1,\ 2(k-1);\ 1, \ldots] \quad (kt>2).$$

If $u=4s$, and k is odd or even, then (2) becomes

$$s\beta^2-x^2(k^2s-1)=1\ldots(3),$$

and the fraction is

$$\sqrt{\frac{k^2s-1}{s}}=[k-1;\ 1,\ 2(ks-1),\ 1,\ 2(k-1);\ 1, \ldots] \quad (ks>1).$$

If k is even (2) reduces to the type (3), u arbitrary. For the few numerical values of k and u satisfying $k^2u^2-4u<4^2$, the Gaussian theory of quadratic forms may be used. The other exceptional cases of $k\leqq 4$, $kt\leqq 2$, $ks=1$, may be disposed of without difficulty. As an example, applying the Serret theorem to (2) for u and k odd, the development of $\sqrt{[(k^2u-4)/u]}$ shows that the third and eleventh complete quotients are the only ones in the period having the denominator 4. Put $k=5$, $u=3$, in (2), then using the convergent immediately preceding the eleventh quotient,

$$\beta/x=(4,\ 1,\ 6,\ 2,\ 1,\ 1,\ 28,\ 1,\ 1,\ 2)=\tfrac{26275}{5401},$$

and we may verify that $3\times 26275^2-71\times 5401^2=4$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

298. **Proposed by W. J. GREENSTREET, Marling School, Stroud, England.**

Find an approximation to the difference between the sums of n harmonic and n arithmetic means between a and b, when a is very nearly equal to b.

299. **Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.**

The sides of a triangle and the area are in arithmetical progression. Find their values, and show there is only *one* solution in rational integers.

GEOMETRY.

331. **Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.**

The center of two spheres radii r_1, r_2, are at the extremities of a straight line $2a$ on which as a diameter a circle is described. Find a point on the circumference from which the greatest portion of spherical surface is visible.

CALCULUS.

255. **Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.**

Find the general values of u and v in terms of x, which satisfy the equations $u^2+l^2(du/dx)^2=v^2$, $u^2+m^2(du/dx)^2=v^2+n^2(dv/dx)^2$

256. **Proposed by S. A. COREY, Hiteman, Iowa.**

Prove that $\sum\limits_{x=0}^{x=\infty}\dfrac{x^{2m}}{1+x^{2n}}=\dfrac{\pi}{2n\sin\dfrac{2m+1}{2n}\pi}$, m and n being positive integers of which n is the greater.

257. **Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.**

$$\text{If } A=\int_0^\infty \frac{dx}{\sqrt{x}\,(2a+x)^n},\quad B=\int_0^\infty \frac{y^n dy}{\sqrt{y}\,(a^2+y^2)^n},\ \text{find } A/B.$$

MECHANICS.

215. **Proposed by R. D. CARMICHAEL, Anniston, Ala.**

Determine the curve in a vertical plane along a chord of which a particle will slide under the force of gravity and the retardation of friction so that it will traverse the whole length of the chord in a time t which is independent of its direction as long as the upper end of the chord remains fixed. Discuss the result.

NOTES AND NEWS.

Professor Heinrich Maschke, of the University of Chicago, died March 1. 1908, after an illness of a few days. He was born in Breslau, Germany, in 1853, and received his education in the universities of Breslau, Berlin, and Goettingen, obtaining his Doctorate at Goettingen in 1880. For ten years he taught in the Guisenstaedische Gymnasium in Berlin, and came to the University of Chicago as Assistant Professor of Mathematics in 1892, was promoted to the associate professorship in 1896, and to the full professorship in 1906. He was widely known both in this country and abroad for his publications in the leading mathematical journals, and was universally beloved by the hundreds of students who have been under his instruction at the University of Chicago. He had been vice-president of the American Mathematical Society and was a member of the Deutsche Mathematiker-Vereinegung. S.

BOOKS.

Scrapbook of Elementary Mathematics. Notes, Recreations, Essays. By William F. White, Ph. D., State Normal School, New Paltz, New York. 12mo. 248 pages, cloth, gilt top. Price, $1.00 net. Chicago: The Open Court Publishing Co.

This book contains a collection of Essays, Recreations, and Notes on Mathematical Topics, presented in a somewhat humorous style, but not in any way sacrificing accuracy of statement. Many interesting topics are discussed. We give a few of the thirty or more subjects: A Few Numerical Curiosities, Algebraic Fallacies, Geometrical Puzzles, The Three Famous Problems of Antiquity, Mathematical Symbols, A Few Surprising Facts in the History of Mathematics, Magic Squares, Quotations on Mathematics, The Golden Age of Mathematics, Alice in the Wonderland of Mathematics, etc. Some topics are perhaps too briefly treated, yet enough is given to arouse the interest of students of mathematics. F.

Graphic Algebra. By Arthur Schultze, Ph. D., Assistant Professor of Mathematics, New York University, Head of the Department of Mathematics, High School of Commerce. 8vo. Cloth, viii + 93 pages. Price, 80 cents net. New York: The Macmillan Co.

This little book contains a fine collection of problems and many valuable suggestions. The graphic solutions of the Quadratic and Cubic Equations, as well as other equations, are well illustrated. F.

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THE GALOIS GROUP OF A RECIPROCAL QUARTIC EQUATION.

By DR. L. E. DICKSON, The University of Chicago.

1. The Galois group of a given algebraic equation, $E=0$, gives complete information as to its solvability by radicals, the number and degree of independent root extractions necessary in case the equation is solvable, and reflects the possible factorizations of E within the chosen domain of rationality, R. The Galois group, G, is characterized by the properties that

(i) Every rational function of the roots which remains (numerically) unaltered by all the substitutions of G equals a quantity in the domain R;

(ii) Every rational function of the roots which equals a quantity in R remains (numerically) unaltered by all the substitutions of G.

Throughout the discussion, the coefficients of every rational function considered are assumed to belong to the domain R.

In determining the group, G, for a given equation, one usually resorts to more or less fortunate guesses as to the rational functions to be employed with success. The present treatment for

$$(1) \qquad E\equiv x^4-ax^3+bx^2-ax+1=0$$

is straightforward and leads in a natural and pleasing way to the criteria desired. While the domain, R, chosen is an arbitrary domain containing a and b, we shall speak of numbers in R as rational, numbers not in R as irrational, and the reader may, if he prefers concreteness, take R to be the domain of all rational (integral or fractional) numbers.

The Galois group for (1) may be any one of ten groups*; necessary and sufficient conditions on a and b for a specified group are determined, and a summary given in § 8.

As the Galois theory is applied only to equations† with distinct roots, we assume that the discriminant $\triangle$ of (1) is not zero. By the usual formula, or more simply by § 4, we find that

*The discussion in the MONTHLY, 1904, p. 195, is incomplete in several respects.

†Multiple factors are first to be determined (by the greatest common divisor process, for example) and removed, as may be done in the domain, R.

$$(2) \qquad \triangle = 4AB^2, \ A = (1+\tfrac{1}{2}b)^2 - a^2, \ B = a^2 - 4b + 8.$$

Hence we assume throughout the paper that $A \neq 0$, $B \neq 0$.

2. Let the notation for the reciprocal roots be chosen so that

$$(3) \qquad x_1 x_2 = 1, \ x_3 x_4 = 1.$$

If either i or j has a value 1 or 2, the equality $x_i x_j = x_1 x_2$ would imply that two x's were equal, contrary to hypothesis. Hence the only products of two roots equal to unity are $x_1 x_2$ and $x_3 x_4$. But $x_1 x_2$ has a rational value. Hence by property (*ii*) of § 1, each substitution of the Galois group, G, either leaves $x_1 x_2$ unaltered formally or else replaces it by $x_3 x_4$. *Hence G is a subgroup of*

$$(4) \quad G_8 : I, \ (12), \ (34), \ (12)(34), \ (13)(24), \ (14)(23), \ (1324), \ (1423).$$

3. We assume for the present that the equation (1) is irreducible in R, the necessary and sufficient conditions on a, b being given in § 5. Then its group is transitive* and hence either G_8 or one of the following:

$$(5) \qquad G_4 : I, \ (12)(34), \ (13)(24), \ (14)(23);$$
$$(6) \qquad C_4 : I, \ (1324), \ (12)(34). \ (1423).$$

We proceed to determine directly the conditions under which the group is G_4 or C_4. For the latter groups, the identity alone leaves x_1 unaltered, so that (by Lagrange's Theorem) any rational function of the roots equals a rational function of x_1. As x_2 is already expressed as such a function by (3), we seek expressions for x_3 and x_4 as rational functions of x_1. Now

$$(7) \qquad x_3 + 1/x_3 = a - v, \ v \equiv x_1 + 1/x_1,$$
$$(8) \qquad (x_3 - 1/x_3)^2 = F \equiv (a-v)^2 - 4.$$

Our aim is therefore to determine the conditions under which F is the square of a rational function of x_1 with coefficients in R:

$$(9) \qquad F = W^2(x_1), \ x_3 - 1/x_3 = W(x_1).$$

To the latter relation we may apply the Galois substitution (12)(34), occurring in both groups (5), (6), and obtain the valid relation

$$x_4 - 1/x_4 = W(x_2), \ \text{or} \ 1/x_3 - x_3 = W(1/x_1).$$

*That is, contains substitutions replacing x_1 by each of the x's.

Hence by addition,

(10) $$W(x_1)+W(1/x_1)=0.$$

For the moment, set $V(x_1)=W(x_1)\div(x_1-1/x_1)$. Then $V(1/x_1)=V(x_1)$. Hence $V(x_1)$ equals a rational function of $v=x_1+1/x_1$. But, by (1),

(11) $$v^2-av+b-2=0.$$

The latter is irreducible in R since (1) is irreducible. Hence any rational function of v can be reduced to a linear function. Thus $V(x_1)$ may be given the form $lv+m$, so that

(12) $$W(x_1)=(x_1-1/x_1)(lv+m).$$

The square of the first factor equals v^2-4. Applying (11), we may express W^2 and F, given by (8), as linear functions of v. The latter must be identically equal since (11) is irreducible. Hence

(13) $$l^2(a^3-2ab)+2lm(a^2-b-2)+m^2a=-a,$$
(14) $$l^2(2a^2-a^2b+b^2-4)+2lm(2a-ab)-m^2(b+2)=a^2-b-2.$$

Replacing x_1 by x_3 in (12), we get

$$W(x_3)=(x_3-1/x_3)[l(x_3+1/x_3)+m].$$

Applying (7), (9_2), and (12) to the latter, we get

(15) $$W(x_3)=(x_1-1/x_1)(lv+m)(la+m-lv).$$

First, let the group be C_4. Apply (1324) to (9_2). Thus

(16) $$x_2-1/x_2=W(x_3),\text{ or } 1/x_1-x_1=W(x_3).$$

Inserting the latter value of $W(x_3)$ in (15), we get

$$(lv+m)(la+m-lv)=-1.$$

Eliminating v^2 by means of (11), we get

(17) $$l^2(b-2)+lma+m^2=-1.$$

The determinant of the coefficients of l^2, lm, m^2 in (13), (14), (17) equals $-4AB$ and hence is not zero (§ 1). The unique set of solutions is

$$l^2=\frac{a^2}{4\,AB},\ lm=\frac{-a\alpha\beta}{4\,AB},\ m^2=\frac{\alpha^2\ \beta^2}{4\,AB},$$
$$\alpha\equiv a^2-2b-4,\ \beta\equiv a^2-3b+2.$$

Since l and m are rational, $\sqrt{(AB)}$ must be rational. This necessary condition for the group C_4 is also sufficient. Indeed, we get

$$l=\tfrac{1}{2}\,\alpha/\sqrt{(AB)},\ m=-\tfrac{1}{2}a\beta/\sqrt{(AB)},$$

and all the roots are rational functions of x_1:

$$(18)\qquad x_3=\tfrac{1}{2}[a-x_1-\frac{1}{x_1}+W(x_1)],\ x_4=\tfrac{1}{2}[a-x_1-\frac{1}{x_1}-W(x_1)],$$

$$(19)\quad W(x_1)=\frac{1}{2\sqrt{(AB)}}(x_1-\frac{1}{x_1})\,[(a^2-2b-4)\,(x_1+\frac{1}{x_1})-a^3+3ab-2a].$$

The Galois group, for a domain R, of an irreducible equation (1) *is the cyclic group C_4 if, and only if, $\sqrt{(AB)}$ is rational in R.*

Next, let the group be G_4. Apply (13)(24) to (9_2). Then

$$x_1-1/x_1=W(x_3).$$

Since this result differs only in sign from (16_2), we get

$$(20)\qquad l^2(b-2)+lma+m^2=+1.$$

The unique set of solutions of (13), (14), (20) is now

$$l^2=\frac{a^2}{4A},\ lm=\frac{a(2+b-a^2)}{4A},\ m^2=\frac{(2+b-a^2)^2}{4A}.$$

Hence $\sqrt{A}$ must be rational. Formula (18), with the suitable value of $W(x_1)$ inserted, expresses x_3 as a rational function of x_1.

The Galois group, for a domain R, of an irreducible equation (1) *is the group G_4 if, and only if, $\sqrt{A}$ is rational in R.*

4. The last result may be readily verified in various ways. For

$$(21)\qquad y_2=x_1x_3+x_2x_4,\ y_3=x_1x_4+x_2x_3,$$

we have, by (3) and the equation (1),

$$y_2+y_3=b-2,\ y_2y_3=x_1^2+x_2^2+x_3^2+x_4^2=a^2-2b.$$

Hence* y_2 and y_3 are the roots of

$$y^2-y(b-2)+a^2-2b=0, \tag{22}$$

$$y_2=\tfrac{1}{2}b-1+\sqrt{A},\quad y_3=\tfrac{1}{2}b-1-\sqrt{A}. \tag{23}$$

Also, if we set $y_1=x_1x_2+x_3x_4=2$,

$$(y_2-y_3)^2=4A,\quad (y_2-2)(y_3-2)=B,\quad \Pi(x_i-x_j)^2=\Pi(y_i-y_j)^2=4AB^2.$$

In particular the three y's are distinct numerically. If $\sqrt{A}$ is irrational, the group cannot be G_4, which leaves y_2 and y_3 unaltered. If $\sqrt{A}$ is rational, the group must be contained in G_4, and being transitive is G_4.

Again, by considering the alternating function $\Pi(x_i-x_j)$, we may conclude that the group is contained in G_4 if, and only if, $\sqrt{\Delta}$ and hence also $\sqrt{A}$ is rational.

5. Consider the question of the reducibility of equation (1) in a domain R. A rational root r, other than ± 1, would imply a second root $1/r$ and hence a quadratic factor. This is also the case for $r=\pm 1$ as the following argument shows. Suppose that E could have a linear and an irreducible cubic factor:

$$E\equiv(x-r)(x^3+\alpha x^3+\beta x-1/r),\quad r,\ \alpha,\ \beta \text{ rational.}$$

Change x to $1/x$ and multiply the new identity by x^4; then

$$E=(1-rx)(1+\alpha x+\beta x^2-\frac{1}{r}x^3),$$

$$E=(x-\frac{1}{r})(x^3-r\beta x^2-r\alpha x-r).$$

Comparing this with the first expression for E, we get

$$r^2=1,\quad \beta=-r\alpha.$$

The cubic factor would then vanish for $x=r$ and be reducible.

Hence† if E is reducible in R, we may set

$$E\equiv(x^2+px+r)(x^2+qx+1/r), \tag{24}$$

$$p+q=-a,\quad p/r+rq=-a,\quad r+1/r+pq=b. \tag{25}$$

If $r=1$, p and q are rational if and only if B, given by (2), is the square of a rational number, $(p-q)^2$.

---

*Or directly from the resolvent cubic of (1), one root being $y_1=2$.

†From this point on we might use Ferrari's three factorizations of a quartic.

If $r=-1$, then $a=0$ and $\sqrt{(-b-2)}$ must be rational.

Finally, let $r^2\neq 1$. The first two equations (25) give

$$p=\frac{-ar}{r+1},\quad q=\frac{-a}{r+1}.$$

For these values, the third condition (25) becomes

$$(26)\qquad y+\frac{a^2}{y+2}=b,\quad y\equiv r+\frac{1}{r}.$$

The former is identical with (22) and has the roots (23). For each y, there are two r's whose rationality depends upon the values of

$$(27)\qquad y_2^2-4=C_+,\quad y_3^2-4=C_-,$$
$$(28)\qquad C_\pm=\tfrac{1}{2}b^2-2-a^2\pm(b-2)\sqrt{A},\quad C_+.C_-=a^2B.$$

If $a\neq 0$, no one of the four values of r is ± 1 (values not falling under the case being treated). If $a=0$, then $C_-=0$ and we have only to consider the two r's connected with $C_+=b^2-4$.

Theorem. *The necessary and sufficient conditions that* (1) *be irreducible in a domain R are that $\sqrt{B}$ be irrational; that either $\sqrt{A}$ be irrational or, in the contrary case, $\sqrt{C_+}$ and $\sqrt{C_-}$ both irrational if $a\neq 0$, $\sqrt{C_+}$ and $\sqrt{(-b-2)}$ both irrational if $a=0$.*

For the case $a=0$, the conditions reduce to the irrationality of

$$\sqrt{(-b+2)},\quad \sqrt{(-b-2)},\quad \sqrt{(b^2-4)}.$$

6. Having determined the Galois group of an irreducible equation (1), we pass to the reducible case. First, let B be the square of a number t of the domain R. We may set

$$(29)\qquad x_1+x_2-x_3-x_4=t.$$

Since $t^2=B\neq 0$ (§ 1), the group is a subgroup of

$$(30)\qquad H_4 : I,\ (12),\ (34),\ (12)(34),$$

the remaining substitutions in (4) replacing t by $-t$. Now

$$x_1+x_2=\tfrac{1}{2}(a+t),\quad x_3+x_4=\tfrac{1}{2}(a-t),$$

so that the rationality of the x's depends upon

(31) $$D_{\pm}=(a\pm\sqrt{B})^2-16,\ \ D_+.D_-=64A.$$

If $\sqrt{D_+}$ is rational and $\sqrt{D_-}$ is irrational, then x_1 and x_2 alone are rational, and the group is I, (34). If $\sqrt{D_{\pm}}$ are both irrational and $\sqrt{A}$ rational the group is the common subgroup of H_4, G_4 (see § 8).

7. Finally, let $\sqrt{B}$ be irrational, $\sqrt{A}$ rational, and just one of the pair $\sqrt{C_{\pm}}$ be rational ($a\neq 0$), one of the pair $\sqrt{C_+}$, $\sqrt{(-b-2)}$ rational ($a=0$). Since $\sqrt{A}$ is rational, the group is a subgroup of G_4. Since $\sqrt{B}$ is irrational the group is not contained in [I, (12) (34)], by (29). By (27),

$$C_+=(x_1x_3-x_2x_4)^2,\ \ C_-=(x_1x_4-x_2x_3)^2.$$

Hence, for $a\neq 0$, the group is composed of the identity and (14) (23) or (13) (24) according as $\sqrt{C_+}$ or $\sqrt{C_-}$ is irrational.

For $a=0$, $y_2=b$, $y_3=-2$. Set $\alpha=x_1+x_4$. Then

$$-\alpha^2=(x_1+x_4)(x_2+x_3)=2+y_2,\ \ \alpha=\sqrt{(-b-2)}.$$

If α is irrational, the group is I, (13) (24). If α is rational, then $\alpha\sqrt{B}=2\sqrt{(b^2-4)}$ is irrational ($\alpha=0$ would imply $y_2=y_3$, $\Delta=0$), and hence $x_1x_3-x_2x_4$ is irrational, so that the group is I, (14) (23).

8. The following tabular summary gives the necessary and sufficient conditions on the coefficients a and b of the reciprocal quartic (1) that its Galois group for a domain R will be a specified one of the ten subgroups of G_8. For the abbreviations A, B, C, D, see (2), (28), (31); for the first four groups, see (4), (5), (6), (30).

G_8 : $\sqrt{A}$, $\sqrt{B}$, $\sqrt{(AB)}$ all irrational.
C_4 : $\sqrt{B}$ irrational, $\sqrt{(AB)}$ rational.
G_4 : $\sqrt{B}$ irrational, $\sqrt{A}$ rational, $\sqrt{C_{\pm}}$ both irrational if $a\neq 0$;
$\sqrt{C_+}\equiv\sqrt{(b^2-4)}$ and $\sqrt{(-b-2)}$ both irrational if $a=0$.
H_4 : $\sqrt{B}$ rational, $\sqrt{D_{\pm}}$ and $\sqrt{A}$ all irrational.
{I, (12) (34)} : $\sqrt{B}$ and $\sqrt{A}$ rational, $\sqrt{D_{\pm}}$ both irrational.
{I, (12)} : $\sqrt{B}$ and $\sqrt{D_-}$ rational, $\sqrt{D_+}$ irrational.
{I, (34)} : $\sqrt{B}$ and $\sqrt{D_+}$ rational, $\sqrt{D_-}$ irrational.
{I} : $\sqrt{B}$ and $\sqrt{D_{\pm}}$ all rational.
{I, (13) (24)} : $\left\{\begin{array}{l}\sqrt{B}\text{ and }\sqrt{C_-}\text{ irrational, }\sqrt{A}\text{ rational }(a\neq 0);\\ \sqrt{B}\text{ and }\sqrt{(-b-2)}\text{ irrational }(a=0).\end{array}\right.$
{I, (14) (23)} : $\left\{\begin{array}{l}\sqrt{B}\text{ and }\sqrt{C_+}\text{ irrational, }\sqrt{A}\text{ rational }(a\neq 0);\\ \sqrt{B}\text{ irrational, }\sqrt{(-b-2)}\text{ rational }(a=0).\end{array}\right.$

POSTSCRIPT. After my paper had been printed, I noticed that the discussion on pp. 73-74 could be materially simplified by introducing instead of V the function $U(x_1)=W(x_1).(x_1-1/x_1)$. Then $U=Lv+M$. The

change amounts to throwing the first factor in (12) into the denominator. In place of (13) and (14) we get

$$L(La+2M)=0,\ L^2(2-b)+M^2=4A.$$

According as the group is C_4 or G_4, we have

$$L(a-v)+M=\mp(Lv+M).$$

Hence for C_4, $La+2M=0$, $L^2B=16A$, so that L and M are rational if and only if $\sqrt{(AB)}$ is rational. For G_4, we have $L=0$, $M^2=4A$, so that $\sqrt{A}$ must be rational. In either case the expression for $W(x_1)$ has x_1-1/x_1 in the denominator, but is much shorter than that in the paper.

ON CERTAIN TRANSCENDENTAL FUNCTIONS DEFINED BY A SYMBOLIC EQUATION.*

By PROFESSOR R. D. CARMICHAEL, Anniston, Alabama.

1. Introduction and Definition.

The object of this paper is to define a certain large class of transcendental functions of real variables to which the previous consideration of a geometric subject† gave rise and to point out a few of their general properties. Attention will be confined chiefly to questions of continuity and development in series.

We shall take

(1) $$u=f(x) \text{ and } v=f(y),$$

and put

(2) $$u^v=v^u;$$

thus we have an equation which is satisfied identically when x is put for y; and therefore one solution of the symbolic equation (2) is $y=x$, whatever function is denoted by f. We discard at once this solution as not pertinent to the present discussion.

In order to obtain a second solution, put

(3) $$v=\mu u,$$

*Read before the American Mathematical Society, September 5, 1907.

†Carmichael, The American Mathematical Monthly, Vol. XIII, pp. 221-226, 1906.

where μ is another variable whose law of variation is determined conjointly by equations (2) and (3). A substitution of the latter value of v in the former equation will yield, by an easy reduction,

$$u=f(x)=\mu^{1/(\mu-1)}, \tag{4}$$
$$v=f(y)=\mu^{\mu/(\mu-1)}. \tag{5}$$

Now suppose that the functional equations (4) and (5) are solved, giving

$$x=g(\mu^{1/(\mu-1)}), \tag{6}$$
$$y=g(\mu^{\mu/(\mu-1)}), \tag{7}$$

where g is the function inverse to f. Equations (6) and (7) define y as a function of x; this is the general function to be studied.*

It should be noticed that the preceding discussion does not show that the two given solutions of (2) are the only ones possible. On the other hand it seems indeed quite probable that such a transcendental equation may have other sets of real solutions; and these may give rise to other quite as interesting functions as those of this paper.

2. Continuities and Discontinuities of the Expressions.

In the present section we shall find the conditions under which $\mu^{1/(\mu-1)}$ and $\mu^{\mu/(\mu-1)}=\mu.\mu^{1/(\mu-1)}$ are continuous. It is evident that they are both continuous only when μ itself is continuous.

We shall find that $\mu^{1/(\mu-1)}$ has a continuous graph in the first quadrant. It will now be shown that its graph is discontinuous in the second, third and fourth quadrants; and the nature of the discontinuity will be exhibited.

In the second and third quadrants μ is negative. Consider the value $\mu=-p/q$, where p/q is a fraction in its lowest terms and p and q are both positive. At most one of them can be even. Then

$$u=\mu^{1/(\mu-1)}=\left(-\frac{p}{q}\right)^{\frac{-q}{p+q}}=\left(-\frac{q}{p}\right)^{\frac{q}{p+q}}. \tag{8}$$

Now u is imaginary if both p and q are odd; for then it is equal to an even root of a negative number. And u is real if one of the numbers p and q is odd and the other even. Now consider the five values of μ:

$$-\frac{(2n+1)\,p-2}{(2n+1)q},\quad -\frac{(2n+1)\,p-1}{(2n+1)q},\quad -\frac{p}{q},\quad -\frac{(2n+1)\,p+1}{(2n+1)q},\quad -\frac{(2n+1)\,p+2}{(2n+1)q},$$

*The curve $u^v=v^u$ was studied by Dan Bernoulli and by Euler. The latter gave a parametric representation similar to that here employed, and plotted the curve in the first quadrant. (See Euler, *Introductio*, Vol. 2, p. 294, edition of 1748.) A study from the viewpoint of function theory is thought to be of sufficient importance to justify this presentation. The function well illustrates several important conceptions of the general theory.

where n is a positive integer and one of the numbers p, q is odd and the other even. If p is even, then in the second and fourth fractions, both numerator and denominator are odd, and u is imaginary for each of these values; but to the first and fifth values there evidently correspond real points of the locus. By taking n large at pleasure the first, second, fourth, and fifth values of μ can be brought indefinitely near its third value; and hence on each side of the point corresponding to $\mu=-p/q$ is a point infinitely near to it in the domain of μ, such that to this point corresponds no point on the locus. Moreover this point is seen to lie between two others infinitely near each other such that to each of these corresponds a point of the locus.

If q is even we may consider the set of values

$$-\frac{(2n+1)p}{(2n+1)q+2},\ -\frac{(2n+1)p}{(2n+1)q+1},\ -\frac{p}{q},\ -\frac{(2n+1)p}{(2n+1)q-1},\ -\frac{(2n+1)p}{(2n+1)q-2}$$

in which p is necessarily odd. A discussion will yield the same conclusion as that obtained when p is even.

These discussions lead readily to the following theorems:

When μ is negative there is in every interval, however small, an infinite number of points on the graph while at the same time there are in the same interval in infinite number of discontinuities. Further, it is clear that *in any such interval $\mu^{1/(\mu-1)}$ is dense, while the points of discontinuity also constitute a dense aggregate.* Finally, *for each point of discontinuity arising as above the function $\mu^{1/(\mu-1)}$ is imaginary.*

In the fourth quadrant u is negative while μ is positive. A discussion similar to that above will lead to the conclusion that *in the fourth quadrant the graph of $\mu^{1/(\mu-1)}$ is infinitely discontinuous.* The discontinuity here arises not by u taking on imaginary values, but by its taking (when real) only positive values for values of μ infinitely near to any value of μ which corresponds to a real point in the fourth quadrant. This is readily seen from a discussion of the following five values of μ:

$$\frac{(2n+1)p-2}{(2n+1)q},\ \frac{(2n+1)p-1}{(2n+1)q},\ \frac{p}{q},\ \frac{(2n+1)p+1}{(2n+1)q},\ \frac{(2n+1)p+2}{(2n+1)q}.$$

The reader can now readily supply the discussion requisite to establish the fact of infinite discontinuity in the present instance.

The different ways in which the discontinuity arises in these two cases is interesting. A more detailed discussion would show that in the third quadrant the discontinuity arises from the introduction both of imaginary values and of such as these just referred to in the discussion of the locus in the fourth quadrant. It is thence easy to show that along the same branch of the curve in the third quadrant there are now three dense aggre-

gates: (1) the aggregate of real points of the locus; (2) the aggregate of points of discontinuity corresponding to those values of μ for which $\mu^{1/(\mu-1)}$ becomes imaginary; (3) the aggregate of points of discontinuity corresponding to those values of μ for each of which $\mu^{1/(\mu-1)}$ takes on a real positive value but no negative value.

From the preceding discussion it follows that if the graph of $\mu^{1/(\mu-1)}$ possesses a continuous branch it lies in the first quadrant. We proceed to establish the fact of its continuity here.

Let a be any value of μ except $\mu=1$, and let the vicinity of a be the interval $a-h$ to $a+h$. Represent $\mu^{1/(\mu-1)}$ by $F(\mu)$. Then the well known necessary and sufficient condition of continuity that for each $e>0$ there must exist an h such that

$$| F(a+h)-F(a-h) | < e$$

is evidently fulfilled. Hence $F(\mu)$ is continuous everywhere except possibly at $\mu=1$. But as $\mu \doteq 1$ from either side,

$$\mu^{1/(\mu-1)}=[1+(\mu-1)]^{1/(\mu-1)} \doteq e=2.712..,;$$

and therefore it follows that the function is continuous at $\mu=1$. Hence it is everywhere continuous. That is, its locus has a continuous branch. As we have seen, this must be in the first quadrant.

Since $\mu^{\mu/(\mu-1)}=\mu.\mu^{1/(\mu-1)}$, the question of the continuity of this function does not require separate consideration; for, since μ is to be taken continuous, it is evident that $\mu^{\mu/(\mu-1)}$ is continuous for all values of μ for which $\mu^{1/(\mu-1)}$ is continuous.

Since $u=\mu^{1/(\mu-1)}$ and $v=\mu^{\mu/(\mu-1)}$, the preceding discussion leads readily to the conclusion that v is a continuous function of u at every point for which both u and v are positive, and equation (2) is satisfied. Likewise u is a continuous function of v under the same limitations. We may then have the theorem:

u and v each is a continuous function of the other throughout the domain of positive rational and irrational numbers.

Now, if in equations (6) and (7) the function denoted by g is continuous, the preceding discussion yields readily the following:

y is a continuous function of x throughout the domain represented by $\mu^{1/(\mu-1)}$ when the domain of μ is all positive rational or irrational numbers.

3. Development in Series.

Resuming equation (4),

$$f(x)=u=\mu^{1/(\mu-1)}, \tag{9}$$

we shall first express μ as an infinite series in terms of u. This will be carried out by the aid of Lagrange's formula and through the help of certain simple substitutions, as follows:*

From (9) we have $u^{\mu-1}=\mu$. Let $u=e^z$. Then $e^{z(\mu-1)}=\mu$. Let $z(\mu-1)=t$; then $e^t=\mu=(\mu-1)+1=t/z+1$. Hence,

$$t=-z+ze^t. \tag{10}$$

If $t=a+b\,\phi(t)$, we have by Lagrange's formula,

$$t=a+b\,\phi(a)+\frac{b^2}{2\,!}\frac{d}{da}[\phi(a)]^2+\ldots+\frac{b^n}{n\,!}\left(\frac{d}{d\,a}\right)^{n-1}[\phi(a)]^n+\ldots \tag{11}$$

Here we have by comparison with (10), $\phi(t)=e^t$, $a=-z$, $b=z$; by a substitution of these values equation (11) reduces to

$$t=-z+ze^{-z}+z^2e^{-2z}+\frac{3z^3}{2!}e^{-3z}+\ \ldots\ +\frac{n^{n-2}z^n}{(n-1)\,!}e^{-nz}+\ldots$$

But $\mu=t/z+1$, as we have seen; and therefore equation (9) readily yields

$$\mu=e^{-z}+ze^{-2z}+\frac{3z^2}{2!}e^{-3z}+\ \ldots\ +\frac{n^{n-2}z^{n-1}}{(n-1)\,!}e^{-nz}+\ldots$$

Since $z=\log_e u$ and $e^{-z}=1/u$, this becomes

$$\mu=\frac{1}{u}\left[1+\frac{\log_e u}{u}+\frac{3}{2\,!}\left(\frac{\log_e u}{u}\right)^2+\ldots+\frac{n^{n-2}}{(n-1)\,!}\left(\frac{\log_e u}{u}\right)^{n-1}+\ldots\right],$$

and since $\mu\,u=v$, we have

$$v=1+\frac{\log_e u}{u}+\frac{3}{2\,!}\left(\frac{\log_e u}{u}\right)^2+\ldots+\frac{n^{n-2}}{(n-1)\,!}\left(\frac{\log_e u}{u}\right)^{n-1}+\ldots, \tag{12}$$

a development of v in terms of u. Now since u and v enter equation (2) in just the same way, it is clear that the development of u in terms of v may be obtained simply by an interchange of u and v in (12); in other words, the series in (12) has the interesting property that it reverts into the same series in the other variable. Hence, if we denote this function by T, we have

*Cf. THE AMERICAN MATHEMATICAL MONTHLY, Vol. XIII, pp. 18, 72; 1906.

$$u=T(v),\ \ v=T(u).$$

Therefore, $\qquad v=T[T(v)]=T^2(v).$

But $\qquad v=T^{-1}[T(v)],$

whatever function is denoted by T. Hence the above function T is equal to its inverse function. Therefore we may think of T as a functional operation such that

$$T^2=1,\ \text{but}\ T\neq +1\ \text{or}\ -1.$$

We consider now the question of the convergency of the series in (12). We shall apply the test of the $(n+1)$th term divided by the nth term. To find the limiting value of this quotient, we have the following equations:

$$\frac{(n+1)^{n-1}}{n!}\left(\frac{\log_e u}{u}\right)^n \div \frac{n^{n-2}}{(n-1)\,!}\left(\frac{\log_e u}{u}\right)^{n-1}$$

$$=\left(\frac{n+1}{n}\right)^{n-1}\frac{\log_e u}{u}=\frac{n}{n+1}\left(1+\frac{1}{n}\right)^n\frac{\log_e u}{u}.$$

But $\lim\limits_{n\doteq\infty}\frac{n}{n+1}\left(1+\frac{1}{n}\right)^n=e.$ Hence the above ratio is less than 1 and the series is convergent for every case for which $\frac{\log_e u}{u}<\frac{1}{e}$. Now it is easy to show that $\frac{\log_e u}{u}$ takes on its maximum value when $u=e$ and that this value is $1/e$. Hence the series in (12) is certainly convergent for every value of u except $u=e$; and for this latter value the preceding discussion gives no answer. For this case, however, it may be shown by other means that $v=e$; for $u=\mu^{1/(\mu-1)}$ takes on the value e only when $\mu=1$, μ being confined to real values; and since $v=\mu\, u$, $v=u$ for this particular value.

We append here also an interesting expansion of v in exponential form in terms of u. From $u=\mu^{1/(\mu-1)}$ we have $u^{\mu-1}=\mu$; $u^\mu=\mu\, u$. Whence

$$\mu=\frac{u^\mu}{u}=\frac{u^{\frac{u^\mu}{u}}}{u}$$

Hence $\mu u=v=u^{\frac{u^{\frac{u^{\frac{u\cdots}{u}}}{u}}}{u}}$

Evidently the corresponding expansion of u in terms of v is gotten by an interchange of these two variables in the last equation.

PI IN ASIA.

By PROFESSOR GEORGE BRUCE HALSTED, Greeley, Colorado.

In Tokyo at the Buddhist temple of Sengakuji lie buried the forty-seven Ronin, the national heroes of feudal Japan. Just within the gate, in a two-storied building, swords, armor and other relics of these heroes are shown on payment of a fee. By the side of the path leading to the tombs is a well with the inscription, "Here they washed it." No one in Japan needs to be told that "*it*" was the bloody head they were bringing to the grave of their lord, that dead master for whom they considered it the highest privilege thus to forfeit all their lives. The popular reverence for these heroes is still attested not only by the incense perpetually kept burning before their tombs but in stranger fashion by the fresh visiting cards constantly left upon their graves.

All the world knows their exploit, but who knows that one of them, Shigekiyo Matsumura, was the greatest Asiatic mathematician of his age, who in his work *Sanso*, published in 1663, calculated the length of one side of a regular inscribed polygon of 32768 or 2^{15} sides, obtaining

$$0.000095873798655313483$$

and thence for the value of pi 3.141592648, which is true to seven places of decimals, to eight significant figures, while seven is the greatest number that can be obtained by any measurement whatsoever of the most modern science. He also, among many other things, evaluates the volume of the sphere, and treats fully the problem of magic squares, describing the solution of the case where the first 19^2 numbers are arranged in the form of a square. He also treats an analogous problem, not rediscovered in Europe, which may be called the problem of magic circles. solving the case where the number of diameters and the number of circles are eight and sixteen, respectively.

In Europe in 1585 Adriaan Anthoniszoon, father of Adrian Metius, gave for pi 355/113=3.1415929. But this, the most extraordinary of all fractional values, had come a whole millennium earlier in Asia. Yoshio Mikami, under date of January 13, 1908, writes me:

"I have just finished by translation of your *Rational Geometry* (309 pages). To your historical note on pi I have added the following words:

At the end of the Three Kingdoms Lin Huy of Wei gave for pi 157/50 = 3.14. Two centuries afterwards Tsu Chung-tse (425-499) gave the value 355/113. In Japan Seki Kowa employed the same value of pi as Tsu, and this value was given in a printed book of his day."

ON THE RECIPROCAL QUARTIC EQUATION.

By DR. R. L. BERGER.

In this paper it is proposed to determine the Galois group of the reciprocal quartic equation,

$$x^4-ax^3+bx^2-ax+1=0\ldots(1),$$

for the domain of rationality $R(1)$. We shall establish the conditions for which the group of the equation is transitive, and hence the condition that the equation is irreducible; and consider the possible intransitive groups when these conditions are not fulfilled.

Calling the roots of (1) α_0, α_1, β_0, β_1; α_0 the reciprocal of β_0, and α_1 the reciprocal of β_1,

$$\alpha_0\beta_0=\alpha_1\beta_1=1\ldots(2).$$

If we assume that the roots of (1) are distinct, the Galois group of (1) is a subgroup of the group,

$$G_8\equiv[1;\ (\alpha_0\beta_0);\ (\alpha_1\beta_1);\ (\alpha_0\beta_0)(\alpha_1\beta_1);\ (\alpha_0\alpha_1)(\beta_0\beta_1);$$
$$(\alpha_0\beta_1)(\alpha_1\beta_0);\ (\alpha_0\alpha_1\beta_0\beta_1);\ (\alpha_1\alpha_0\beta_1\beta_0)].$$

For, any substitution not in G_8 leaves the relation (2) unaltered only if the equation (1) has a pair of equal roots. The group G_8 has only two transitive subgroups, viz:

$$C_4\equiv[1;\ (\alpha_0\alpha_1\beta_0\beta_1);\ (\alpha_1\alpha_0\beta_1\beta_0);\ (\alpha_0\beta_0)(\alpha_1\beta_1)],$$
$$\text{and } G_4\equiv[1;\ (\alpha_0\alpha_1)(\beta_0\beta_1);\ (\alpha_0\beta_1)(\alpha_1\beta_0);\ (\alpha_0\beta_0)(\alpha_1\beta_1)].$$

Hence, if we impose such conditions upon the coefficients of (1) that its group is either G_8, G_4, or C_4 we have the necessary and sufficient conditions that the equation be irreducible. For this purpose it is necessary to compute the values in terms of the coefficients of functions belonging to each of the following subgroups of G_8:

$$G_4;\ C_4;\ H_4\equiv[1;\ (\alpha_0\beta_0);\ (\alpha_1\beta_1);\ (\alpha_0\beta_0)(\alpha_1\beta_1)].$$
$$G'_2\equiv[1;\ (\alpha_0\alpha_1)(\beta_0\beta_1)];\ G''_2\equiv[1;\ (\alpha_0\beta_1)(\alpha_1\beta_0)].$$

From the equation (1); $\alpha_0+\beta_0+\alpha_1+\beta_1=a\ldots(3)$,
$(\alpha_0+\beta_0)(\alpha_1+\beta_1)=b-2\ldots(4)$.

By means of (3) and (4) we easily find;

$\phi \equiv (\alpha_0+\beta_0)-(\alpha_1+\beta_1)=\sqrt{[2-b+\frac{1}{4}a^2]}$, belonging to H_4... (5),
$\psi \equiv (\alpha_0-\beta_0)(\alpha_1-\beta_1)=\sqrt{[(1+\frac{1}{2}b)^2-a^2]}$, belonging to G_4... (6),
$\phi.\psi$ belonging to C_4... (7).

With the aid of (5) and (6);

$\phi' \equiv \alpha_0+\alpha_1=\sqrt{\{\frac{1}{4}a^2-(1+\frac{1}{2}b)-\sqrt{[(1+\frac{1}{2}b)^2-a^2]}\}}$, belonging to G'_2... (8),
$\phi'' \equiv \alpha_0+\beta_1=\sqrt{\{\frac{1}{4}a^2-(1+\frac{1}{2}b)+\sqrt{[(1+\frac{1}{2}b)^2-a^2]}\}}$, belonging to G''_2... (9),
$\phi''' \equiv (\alpha_0-\beta_1)^2=\sqrt{(b^2-4)}$ (when $a=0$), belonging to G''_2... (10).

The values for ϕ and ψ are given up to a rational factor; those for ϕ', ϕ'', ϕ''' up to a rational term.

In the study of the cases where the group is intransitive we shall make use of the two additional functions:

$x_0 \equiv \alpha_0-\beta_0=+\sqrt{\{\frac{1}{2}a^2-(b+2)+a\sqrt{[\frac{1}{4}a^2+2-b]}\}}$,
belonging to $H_2 \equiv [1;\ (\alpha_1\beta_1)]$... (11),
$x_1 \equiv \alpha_1-\beta_1=+\sqrt{\{\frac{1}{2}a^2-(b+2)-a\sqrt{[\frac{1}{4}a^2+2-b]}\}}$,
belonging to $H'_2 \equiv [1;\ (\alpha_0\beta_0)]$... (12).

From the definition of the Galois group of an equation for a domain of rationality R it follows that if a rational function of the roots of the equation belonging to a group H has a value not in R the group of the equation is not contained in H. Hence we need to consider subgroups of a group H only when a function belonging to H has its value in R.

The group of (1) will depend on the character of the functions (5)-(12) and we have in the following three cases the conditions for which the group G is transitive:

1) ϕ irrational, ψ irrational, and $\phi.\psi$ irrational.

2) ϕ irrational, ψ irrational, and $\psi.\phi$ rational.

$\therefore G=C_4$ or a subgroup of C_4. But its subgroups are excluded as they are subgroups of H_4 to which ϕ belongs. $\therefore G=C_4$.

3) ϕ irrational, ψ rational. Then $G=G_4$, G'_2 or G''_2.

We distinguish two cases:

I) $a\neq 0$. If ϕ' and ϕ'' are irrational, *i. e.*, if the two values $+\sqrt{\{\frac{1}{4}a^2-(1+\frac{1}{2}b)\pm\sqrt{[(1+\frac{1}{2}b)^2-a^2]}\}}$ are irrational, G'_2 and G''_2 are excluded, and $G=G_4$.

II) $a=0$. In this case $\phi''=0$ and we use $\phi'''=\sqrt{(b^2-4)}\neq 0$ to exclude G'_2. $\phi'''\neq 0$, since $b=2$ makes ϕ rational and $b=-2$ makes the equation 1) have a pair of equal roots. If then ϕ' is irrational, *i. e.*, in this case, $\sqrt{(-2-b)}$ is irrational and if $\sqrt{(b^2-4)}\equiv\phi'''$ is irrational, $G=G_4$.

These are the only cases in which the group G is transitive. Therefore we conclude that the necessary and sufficient conditions for the irreducibility of (1) are, if $a\neq 0$,

I) $\sqrt{(2-b+\frac{1}{4}a^2)}$ irrational, and $\sqrt{[(1+\frac{1}{2}b)^2-a^2]}$ irrational.

II) $\sqrt{(2-b+\frac{1}{4}a^2)}$ irrational, $\sqrt{[(1+\frac{1}{2}b)^2-a^2}$ rational, and $\sqrt{\{\frac{1}{4}a^2-(1+\frac{1}{2}b)\pm\sqrt{[(1+\frac{1}{2}b)^2-a^2]}\}}$ irrational.

III) When $a=0$, $\sqrt{(-2-b)}$, $\sqrt{(2-b)}$, $\sqrt{(b^2-4)}$ all irrational. When these conditions are not fulfilled the equation is reducible and its group may be any of the intransitive subgroups of G_8. By examining the values of the set of functions (5)-(12) for any particular case the groups G can be determined. The following cases arise:

1) ϕ rational, ψ irrational. Hence, $G=H_4$, H_2, or H'_2, dependent upon rational or irrational nature of x_0, x_1. Since ψ is irrational, x_0 and x_1 are not both rational. Therefore $G\neq G_1$.

2) ϕ rational, ψ rational. Hence, $G=G_2\equiv[1;\ (\alpha_0\beta_0)(\alpha_1\beta_1)]$, the greatest common subgroup of G_4, H_4 and C_4, or if all remaining functions are rational, then $G=G_1$.

3) ϕ irrational, ψ rational. Hence, $G=G'_2$ or G''_2, according as ϕ' or ϕ'' is rational. The case ϕ', ϕ'' both irrational has been included in the previous discussion.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

GEOMETRY.

322. Proposed by FRANK LOXLEY GRIFFIN, S. M., Ph. D., Instructor in Mathematics, Williams College.

Find all surfaces such that the normal lengths intercepted by the three coordinate planes are in constant ratios for all points.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

$$\left.\begin{array}{l} n-y+\dfrac{dz}{dy}(s-z)=0, \\ m-x+\dfrac{dz}{dx}(s-z)=0, \end{array}\right\}$$ is the equation to the normal, where (x, y, z) is the point on the surface.

$(x+zdz/dx,\ y+zdz/dy,\ 0)$, where the normal meets the xy plane.

$(x-ydy/dx,\ 0,\ ydy/dz+z)$, where the normal meets the xz plane.

$(0,\ y-xdx/dy,\ xdx/dz+z)$, where the normal meets the yz plane.

Then

$$\pm\frac{x\sqrt{[1+(dx/dy)^2+(dx/dz)^2]}}{y\sqrt{[1+(dy/dx)^2+(dy/dz)^2]}}=a=\pm\frac{xdx}{ydy}\dots(1).$$

$$\pm\frac{x\sqrt{[1+(dx/dy)^2+(dx/dz)^2]}}{z\sqrt{[1+(dz/dx)^2+(dz/dy)^2]}}=b=\pm\frac{xdx}{zdz}\dots(2).$$

$$\pm\frac{y\sqrt{[1+(dy/dx)^2+(dy/dz)^2]}}{z\sqrt{[1+(dz/dx)^2+(dz/dy)^2]}}=c=\pm\frac{ydy}{zdz}\dots(3).$$

$\therefore aydy=xdx,\ bzdz=xdx,\ czdz=ydy,\ aydy+bzdz+czdz=2xdx+ydy.$

$\therefore 2x^2=(b+c)z^2+(a-1)y^2+D.$

$\therefore x^2=Ay^2+Bz^2+D$ are the surfaces satisfying the conditions.

The equation could be put in the form $x^2=ay^2--acz^2+D=0$, where $ac=-b$, as is the case from (1), (2), and (3).

323. Proposed by W. J. GREENSTREET, Marling School, Stroud, England.

S, S' are the foci of two co-vertical parabolas A and B, the axes of which are at right angles. Draw the circle K on SS' as diameter. K is cut in D and E by a straight line parallel to the axis of A such that S' lies midway between it and that axis. Show that the lines $S'D$, $S'E$ are parallel to the two tangents to A which are normals to B.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

The problem as stated is not true.

Let $y^2=4ax$, $x^2=4by$, be the parabolas A, B; $y=mx+a/m$=tangent to A; $y-y'=m(x-x')$ the normal to B.

Since $x'^2=4by'$, $m=-2b/x'$ or $x'=-2b/m$, $y'=x'^2/4b=b/m^2$.

$\therefore y=mx+2b+b/m^2$=normal to B.

But $y=mx+a/m$ and $y=mx+2b+b/m^2$ are the same line.

$\therefore a/m=2b+b/m^2$.

$\therefore m=\frac{1}{4b}[a\pm\sqrt{(a^2-8b^2)}]$, which is true for $a\gtreqless 2\sqrt{2}\,b$.

$x^2-ax+y^2-by=0$ is equation to K, $y=2b$=line parallel to axis of A.

$\therefore x^2-ax+2b^2=0$ or $x=\frac{1}{2}[a\pm\sqrt{(a^2-8b^2)}]$.

$\therefore y=\frac{2bx}{a\pm\sqrt{(a^2-8b^2)}}+b$ are the equations to $S'D$, $S'E$.

$\therefore$ If m were twice as great, $S'D$, $S'E$ would be perpendicular to the two tangents to A which are normals to B.

The lines through S' parallel to the two tangents are given by the equation

$$y=\frac{1}{4b}[a\pm\sqrt{(a^2-8b^2)}]x+b.$$

This line intersects K in the points

$$\left[-\frac{6ab^2 \pm 2b^2 \sqrt{(a^2-8b^2)}}{a^2+4b^2 \pm a\sqrt{(a^2-8b^2)}},\ -\frac{a^2b-12b^3 \pm ab\sqrt{(a^2-8b^2)}}{a^2+4b^2 \pm a\sqrt{(a^2-8b^2)}}\right],$$

the plus and minus signs to be used together.

$$y=\frac{8abx}{a^2+4b^2}+12b^3-a^2b+\frac{48a^2b^3}{a^2+4b^2+a\sqrt{(a^2-8b^2)}}$$

is the line through these points. The tangents to A parallel to $S'D$, $S'E$ are

$$y=\frac{2bx}{a\pm\sqrt{(a^2-8b^2)}}+\frac{a[a\pm\sqrt{(a^2-8b^2)}]}{2b}$$

This line meets $x^2=4by$ in

$$x_1=\frac{4b^2\pm\sqrt{\{16b^4+2a[a\pm\sqrt{(a^2-8b^2)}]^3\}}}{a\pm\sqrt{(a^2-8b^2)}}=r.$$

The tangent at this point makes an angle with the axis of abscissas whose tangent is $x_1/2b$. As this does not equal $-1/m$ the problem, as stated, is not true.

CALCULUS.

252. Proposed by J. H. MEYER, S. J., Augusta, Ga.

Supposing the arc of a semi-circle to be stretched out into a straight line, and an indefinite number of perpendiculars erected on it, each equal to the versed sine of the corresponding arc; what would be the length of the curve traced out by the tops of the perpendiculars?

Solution by CHAS. O. GUNTHER, Acting Professor of Mathematics, Stevens Institute of Technology, Hoboken, N. J.

Assuming a as the radius of the circle, the equation of the curve is $y=\text{vers}\ x/a$, and the required length of the curve is given by the expression

$$s=2\int_0^{\pi a/2}(1+\frac{\sin^2 x/a}{a^2})dx=2\int_0^{\pi a/2}\sqrt{(a^2+1-\cos^2 x/a)}\frac{dx}{a}.$$

Let $\cos x/a=\sin\theta$; then

$$s=2\sqrt{(a^2+1)}\int_0^{\frac{1}{2}\pi}\sqrt{1-\left(\frac{1}{\sqrt{(a^2+1)}}\right)^2\sin^2\theta}\,d\theta=2\sqrt{(a^2+1)}E(\frac{1}{\sqrt{(a^2+1)}},\ \tfrac{1}{2}\pi).$$

Also solved by G. B. M. Zerr.

253. Proposed by PROFESSOR R. D. CARMICHAEL, Anniston, Ala.

Find the maximum number of real points of inflection in each of the quartic curves $y^2=\alpha x^4\pm x^2+\beta$, and find the necessary and sufficient relations between α and β for the existence of this number of points of inflection.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

$y=\sqrt{[\alpha x^4\pm x^2+\beta]}$. Then $\dfrac{dy}{dx}=\dfrac{2\alpha x^3\pm x}{\sqrt{[\alpha x^4\pm x^2+\beta]}}$.

$$\frac{d^2y}{dx^2}=\frac{2\alpha^2x^6\pm3\alpha x^4(2\alpha-1)+6\alpha^2\beta x^2\pm\beta}{(\alpha x^4\pm x^2+\beta)^{\frac{3}{2}}}=0.$$

$$\therefore x^6\pm\frac{3(2\alpha-1)}{2\alpha}x^4+3\beta x^2\pm\frac{\beta}{2\alpha^2}=0.$$

Let $x^2=z$, $\dfrac{3(2\alpha-1)}{2\alpha}=a$, $3\beta=b$, and $\dfrac{\beta}{2\alpha^2}=c$. $\quad\therefore z^3\pm az^2+bz\pm c=0.$

Let $z=u\mp\frac{1}{3}a$. Then $u^3-(\frac{1}{3}a^2-b)u\mp(\dfrac{ab}{3}-\frac{2}{27}a^3-c)=0$, or $u^3-Au\mp B=0$.

$\therefore$ When $A^3/B^2<27/4$, there are three real unequal roots.

$$\therefore\ \frac{[(2\alpha-1)^2-4\alpha^2\beta]^3}{[6\alpha^2\beta(2\alpha-1)-2\alpha\beta-(2\alpha-1)^3]^2}<1.$$

Then $y^2=\alpha x^4+x^2+\beta$ can have six real points of inflection, while $y^2=\alpha x^4-x^2+\beta$ can have but four. The values of α and β can reduce this number in both cases.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

122. Proposed by DR. L. E. DICKSON, The University of Chicago.

If p is prime, $(p^4-1)(p^2-1)$ has no factor of the form $1+p^3x$, $x>0$, if $p>2$; $(p^6-1)(p^4-1)(p^2-1)$ has no factor of the form $1+p^5x$, $x>0$.

Solution by H. S. VANDIVER, Bala, Pa.

I. Assume that $(p^4-1)(p^2-1)$ is divisible by $1+p^3x$ where x is not zero, p *any* integer, not zero. By inspection we see that there exists a positive integer y, not zero, such that

$$(p^4-1)(p^2-1)=(1+p^3x)(1+p^2y).$$

Now write the left hand member in the scale of notation whose radix is p. Then

$$p^5(p-1)+p^4(p-2)+p^3(p-1)+p^2(p-1)+1$$
$$=1+p^3x+p^2y+p^5xy.$$

From which it is evident that $xy \ngtr p-1$, and x and y are each less than p. The right hand member is therefore also expressed in the p-scale of notation, and the two expressions are identical. Whence, equating coefficients of p^4, we obtain $p=2$, and $(2^4-1)(2^2-1)=(1+2^3)(1+2^2)$. Otherwise the assumed equality is impossible. Note that p is not necessarily prime.

II. Assume that $(p^6-1)(p^4-1)(p^2-1)$ has a factor of the form $1+p^5x$, $x>0$, then there is necessarily an integer $y>0$, such that

$$P=(p^6-1)(p^4-1)(p^2-1)=(1+p^5x)(-1+p^2y)\ldots(1).$$

Whence, by expansion and division by p^2,

$$y=1+p^2+p^3x+p^{10}-p^8-p^6-p^5x\ldots(2),$$

and y is therefore of the form $1+p^2+p^3M$.

Since $y>0$, then $M\geqq 0$. If $M=0$, $y=p^2+1$, and from (2),

$$x=p^3-\frac{2p^5}{p^4+p^2-1}.$$

Now p^5 is prime to p^4+p^3-1, and this is contrary to the hypothesis that x is an integer. Assume, then, $M>0$. In this case, evidently $y>p^3$. As a consequence, $x<p^2$, for if $x\geqq p^2$ then from (1), $P>(1+p^7)(-1+p^5)$, and by expanding and dividing by p^2, the result may be written $p^8-1+p^5(p-1)+p^2(p-1)<0$, which is absurd. In a similar way, it may be shown from (1), that $y<p^5$, for all values of x, not zero.

From (2), $y-(1+p^2+p^3x)$ is divisible by p^5. Now since $x<p^2$, $1+p^2+p^3x<p^5$, and since $y<p^5$, we must have $y=p^3x+p^2+1$. Comparing with (2), we obtain

$$p^5-p^3-p=x(p^3x+p^2+1),$$

and evidently $x\equiv 0 \pmod p$. Hence the right hand member is greater than p^5. But the left hand member is less, so (1) is impossible. Note again that p is not necessarily prime.

PROBLEMS FOR SOLUTION.

ALGEBRA.

298. **Proposed by J. F. LAWRENCE, A. B., Professor of Mathematics, Stillwater, Okla.**

If α, β, γ, ..., are the roots of the equation $\sin mx - nx\cos mx = 0$, prove that $\tan^{-1}\frac{x}{\alpha}+\tan^{-1}\frac{x}{\beta}+\ldots+\tan^{-1}\frac{x}{\nu}=0$.

GEOMETRY.

331. **Proposed by DAVID F. KELLEY, New York, N. Y.**

To find the area of a parabolic sector, by a hitherto unpublished method.

CALCULUS.

255. **Proposed by A. H. HOLMES, Brunswick, Maine.**

Evaluate $\int_0^{\frac{1}{2}a}\frac{dx}{\sqrt{[2ax-x^2\sqrt{(a^2-x^2)}]}}$.

256. **Proposed by PROFESSOR R. D. CARMICHAEL, Anniston, Ala.**

Solve the differential equation, $(1+y+2xy)dx+x(1+x)dy=0$.

MECHANICS.

215. **Proposed by HENRY WRITT, Genoa Junction, Wisconsin.**

Suppose two centers of attractive forces A and B having a ratio $1:330,000$, and influence reducing as the second power of the distance, *i. e.*, R^{-2}. Then there is a point, P, on the line joining A and B, where $\frac{1}{AP^2}=\frac{330,000}{BP^2}$, or $1:575$, nearly. At this point the attractions are equal but opposite in direction along AB. It is proposed to find the surface through the point P which is the locus of the direction of the resultant of the two forces directed towards A and B, *i. e.*, the locus of the diagonals of the minimum parallelogram of forces.

Mechanics, and College Algebra. By Professor G. W. Myers, in the College of Education, Pedagogy of Secondary Mathematics, and Pedagogy of Elementary School Mathematics.

BOOKS.

The Theory of Functions of a Real Variable and the Theory of Fourier's Series. By W. E. Hobson, Sc. D., F. R. S., Fellow of Christ's College and Stokes Lecturer in Mathematics in the University of Cambridge. Royal 8vo. Cloth, xvi + 772 pages. Price, $6.50. Published in America by G. P. Putnam's Sons, New York City.

This work presents in a connected form, and renders more easily accessible than hitherto, the chief results of the investigations, in the Theory of Functions of a Real Variable, made by Cantor, Dedekind, Weierstrasse, and others. Gaps in the various theories have been filled up, proofs of many of the theorems have been simplified, and some theorems have been given in a more general form than that in which they were originally discovered.

The table of contents will give some idea of the extensive treatment of some of the most interesting developments in modern mathematics. Thus, Chapter I contains a discussion of Number, and includes a full account of the theories of Real Number, due to Cantor and Dedekind; Chapter II contains an exposition of the theory of sets of points, and includes an account of transfinite cardinal and ordinal Arithmetic; Chapter III deals pretty fully with the general theory of aggregates; Chapter IV discusses the main properties of functions in relation to continuity, discontinuity, etc., including investigations of the properties of important classes of functions; Chapter V is devoted to a discussion of the Foundations of the Integral Calculus, as based upon Riemann's definition of a definite integral, and its extension. Also in this chapter is given an account of the development of the Integral Calculus from the view point of Lebesgue's new definition of the definite integral; Chapter VI is concerned with functions defined as the limits of sequences of functions, and contains an account of the principal properties of functions represented by series, and a discussion of important matters connected with the modes of convergence of series through whole intervals, or in the neighborhood of particular points. An account of the very general results recently obtained by Baire, relating to the representability of functions by means of series, will be found in this Chapter. Chapter VII is devoted to the theory of Fourier's Series.

The work constitutes the most thorough, comprehensive, and satisfactory treatment of the Theory of Functions of Real Variables that has thus far appeared in the English language. B. F. F.

High School Algebra.—Advanced Course. By H. E. Slaught, Ph. D., Associate Professor of Mathematics in the University of Chicago, and N. J. Lennes, Ph. D., Instructor in Mathematics in the Massachusetts Institute of Technology. 12mo. Cloth, vii+194 pages. Price, 75 cents. Chicago: Allyn & Bacon.

It is believed that this text book of Algebra, which includes a review of the authors' Elementary Course, contains all the algebra required by most technical and scientific schools. The material of this course is well selected and skillfully arranged. The graph is used, but not to excess. Some books are making too much of the graph, the danger being that students using a text in which over-emphasis is laid on graphics are likely to go off with the authors on the tangent to the graph. B. F. F.

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ON THE GENERALIZATION OF A THEOREM IN SOLID GEOMETRY.

By ROBERT E. MORITZ, Ph. D., The University of Washington, Seattle.

It is shown in solid geometry that the volume of a truncated right triangular prism is equal to the product of the area of its base by one-third the sum of the lateral edges. In symbols,

$$V_3=B_3\left(\frac{e_1+e_2+e_3}{3}\right), \qquad [1]$$

where B_3 denotes the base; e_1, e_2, e_3 the lateral edges; and V_3 the volume of the truncated prism.

A truncated right quadrangular prism may be divided into two truncated right triangular prisms by passing a plane through a pair of opposite lateral edges, say through e_1 and e_3, where e_1, e_2, e_3, e_4 denote the lateral edges taken in succession. If furthermore B_4 denotes the quadrangular base, B'_3 and B''_3 the respective bases of the component triangular prisms, and V_4 the volume of the truncated quadrangular prism, the application of formula [1] gives

$$V_4=B'_3\left(\frac{e_1+e_2+e_3}{3}\right)+B''_3\left(\frac{e_3+e_4+e_1}{3}\right). \qquad [2]$$

If the base of the truncated prism is a parallelogram, B'_3 and B''_3 are each equal to one-half of B_4, [2] therefore becomes

$$V_4=\frac{B_4}{2}\left(\frac{e_1+e_2+e_3+e_4+e_1+e_3}{3}\right). \qquad [3]$$

But now the upper base is a parallelogram also, and from the fact that the diagonals of this upper base bisect each other it readily follows that

$$e_1+e_3=e_2+e_4,$$

from which

$$e_1+e_3=\frac{e_1+e_2+e_3+e_4}{2}$$

By substituting this value of e_1+e_3, formula [3] reduces to

$$V_4=B_4\left(\frac{e_1+e_2+e_3+e_4}{4}\right), \qquad [4]$$

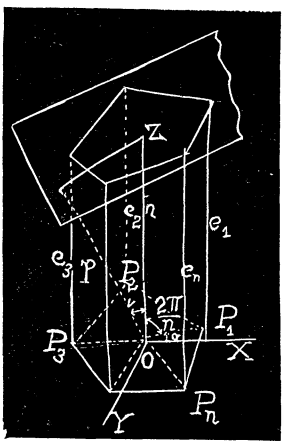
that is,

The volume of a truncated right quadrangular prism, whose base is a parallelogram, is equal to the product of the area of its base by one-fourth the sum of the lateral edges.

The question naturally suggests itself whether this theorem may be generalized so as to apply to a truncated prism of any number of faces. The answer is that it can be, provided the prism is regular. In that case, the corresponding formula for the volume is

$$V_n=B_n\left(\frac{e_1+e_2+e_3+\ldots+e_n}{n}\right). \qquad [5]$$

The writer is not aware that a proof of this theorem has ever been published, nor does the theorem appear to have received even mention in text-books on solid geometry. The following proof is of interest not only as a simple proof of the theorem in question, but also as furnishing an elegant application of a number of trigonometric principles.



Let B_n denote the base of a truncated regular prism of n sides, P_1, P_2, P_3, ..., P_n the vertices of the base, and e_1, e_2, e_3, ..., e_n the corresponding edges.

Through each of the edges and O the center of the base pass planes dividing the solid into n truncated triangular prisms. The bases of these component prisms are equal and each equal to $\frac{B_n}{n}$. Let h represent the edges common to all the triangular prisms. Formula [1] then gives us for the volume of the sum of the truncated triangular prisms,

$$V_n=\frac{B_n}{n}\left(\frac{e_1+e_2+h}{3}+\frac{e_2+e_3+h}{3}+\ldots+\frac{e_n+e_1+h}{3}\right)$$

$$=\frac{B_n}{n}\left[\frac{2(e_1+e_2+e_3+\ldots+e_n)+nh}{3}\right] \qquad [6]$$

Now take the plane of the base of the prism for the xy-plane of coordinates, and a perpendicular at O for the z-axis. Let ρ, α denote the polar coordinates of the vertex P_1 with reference to O as pole and OX as initial line. The central angles subtended by the sides of the polygon are all equal and each equal to $\frac{2\pi}{n}$, hence the rectangular coordinates of the successive vertices are

$$\text{for } P_1,\ x_1=\rho\cos\alpha,\quad y_1=\rho\sin\alpha,$$

$$\text{for } P_2,\ x_2=\rho\cos\left(\alpha+\frac{2\pi}{n}\right),\quad y_2=\rho\sin\left(\alpha+\frac{2\pi}{n}\right),$$

$$\text{for } P_3,\ x_3=\rho\cos\left(\alpha+\frac{4\pi}{n}\right),\quad y_3=\rho\sin\left(\alpha+\frac{4\pi}{n}\right),$$

$$\cdot\qquad\cdot\qquad\cdot\qquad\cdot\qquad\cdot\qquad\cdot$$

$$\text{for } P_n,\ x_n=\rho\cos\left(\alpha+\frac{2(n-1)\pi}{n}\right),\quad y_n=\rho\sin\left(\alpha+\frac{2(n-1)\pi}{n}\right).$$

Let

$$x\cos\lambda+y\cos\mu+z\cos\nu-p=0$$

be the equation of the plane of the upper base of the truncated prism, then

$$z=\frac{p-x\cos\lambda-y\cos\mu}{\cos\nu}$$

The length of any edge e_{k+1} is found by substituting in z for x and y the coordinates x_{k+1}, y_{k+1} of the corresponding vertex of the base; hence,

$$e_{k+1}=\frac{p-\rho\cos\lambda\cos\left(\alpha+\frac{2k\pi}{n}\right)-\rho\cos\mu\sin\left(\alpha+\frac{2k\pi}{n}\right)}{\cos\nu}$$

If we develop $\cos\left(\alpha+\frac{2k\pi}{n}\right)$ and $\sin\left(\alpha+\frac{2k\pi}{n}\right)$ and combine the resulting terms in $\cos\frac{2k\pi}{n}$ and $\sin\frac{2k\pi}{n}$, respectively, we obtain

$$e_{k+1}=\frac{p}{\cos\nu}-\frac{\rho(\cos\alpha\cos\lambda-\sin\alpha\cos\mu)}{\cos\nu}\cos\frac{2k\pi}{n}$$

$$-\frac{\rho(\sin\alpha\cos\lambda-\cos\alpha\cos\mu)}{\cos\nu}\sin\frac{2k\pi}{n}.$$

But p is the perpendicular distance from O to the plane of the upper base and ν is the angle between this perpendicular and OZ or h, hence $\frac{p}{\cos\nu}=h$. Again, the coefficients of $\cos\frac{2k\pi}{n}$ and $\sin\frac{2k\pi}{n}$ do not involve k, they are the same for every edge, hence denoting them by A and B, respectively, we have

$$e_{k+1}=h+A\cos\frac{2k\pi}{n}+B\sin\frac{2k\pi}{n}$$

and we obtain for the sum of all the edges

$$\begin{aligned} e_1+e_2+e_3+\ldots+e_n={}&nh\\ &+A\left(\cos 0+\cos\frac{2\pi}{n}+\cos\frac{4\pi}{n}+\ldots.+\cos\frac{2(n-1)\pi}{n}\right),\\ &+B\left(\sin 0+\sin\frac{2\pi}{n}+\sin\frac{4\pi}{n}+\ldots.+\sin\frac{2(n-1)\pi}{n}\right). \end{aligned} \qquad [7]$$

The series in the parentheses of [7] may be summed, as follows:

$$2\cos 0.\sin\frac{\pi}{n}=\sin\frac{\pi}{n}+\sin\frac{\pi}{n},$$

$$2\cos\frac{2\pi}{n}.\sin\frac{\pi}{n}=\sin\frac{3\pi}{n}-\sin\frac{\pi}{n},$$

$$2\cos\frac{4\pi}{n}.\sin\frac{\pi}{n}=\sin\frac{5\pi}{n}-\sin\frac{3\pi}{n},$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$2\cos\frac{2(n-1)\pi}{n}.\sin\frac{\pi}{n}=\sin\frac{(2n-1)\pi}{n}-\sin\frac{(2n-3)\pi}{n}.$$

Adding,

$$2\sin\frac{\pi}{n}\left(\cos 0+\cos\frac{2\pi}{n}+\cos\frac{4\pi}{n}+\ldots.+\cos\frac{2(n-1)\pi}{n}\right)=\sin\frac{\pi}{n}+\sin\frac{(2n-1)\pi}{n}$$

But

$$\sin\frac{\pi}{n}+\sin\frac{(2n-1)\pi}{n}=2\sin\frac{2n\pi}{2n}\cos\frac{(2n-2)\pi}{2n}=0,$$

since $\sin\frac{2n\pi}{2n}=\sin\pi=0,$

hence

$$\cos 0+\cos\frac{2\pi}{n}+\cos\frac{4\pi}{n}+\ldots.+\cos\frac{2(n-1)\pi}{n}=0. \qquad [8]$$

Again,

$$2\sin 0\,.\sin\frac{\pi}{n}=0,$$

$$2\sin\frac{2\pi}{n}.\ \sin\frac{\pi}{n}=\cos\frac{\pi}{n}-\cos\frac{3\pi}{n},$$

$$2\sin\frac{4\pi}{n}.\ \sin\frac{\pi}{n}=\cos\frac{3\pi}{n}-\cos\frac{5\pi}{n},$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$2\sin\frac{2(n-1)\pi}{n}.\ \sin\frac{\pi}{n}=\cos\frac{(2n-3)\pi}{n}-\cos\frac{(2n-1)\pi}{n}.$$

Adding,

$$2\sin\frac{\pi}{n}\left(\sin 0+\sin\frac{2\pi}{n}+\sin\frac{4\pi}{n}+\ldots+\sin\frac{2(n-1)\pi}{n}\right)$$

$$=\cos\frac{\pi}{n}-\cos\frac{(2n-1)\pi}{n}=2\sin\pi\sin\frac{(n-1)\pi}{n}=0,$$

hence, also,

$$\sin 0+\sin\frac{2\pi}{n}+\sin\frac{4\pi}{n}+\ldots+\sin\frac{2(n-1)\pi}{n}=0. \qquad [9]$$

With the values [8] and [9] equation [7] becomes

$$e_1+e_2+e_3+..,+e_n=nh,$$

and this value of nh put in [6] gives

$$V_n=B_n\left(\frac{e_1+e_2+e_3+\ldots+e_n}{n}\right),$$

that is,

The volume of any truncated regular prism is equal to the product of the area of its base by the arithmetic mean of the edges.

As a limiting case of this theorem we obtain the well known rule for finding the volume of a truncated right circular cylinder.

The above proof could have been somewhat shortened, by choosing the x-axis so as to pass through one of the vertices of the base of the truncated prism. In that case $\alpha=0$ from the outset.

TWO THEOREMS IN THE GEOMETRY OF CONTINUOUSLY TURNING CURVES.

By PROFESSOR O. D. KELLOGG, University of Missouri.

The simple theorems given in the following pages were found necessary in investigating the continuity of certain line integrals occurring in the theory of logarithmic potentials.* They are thought to be interesting examples of the arithmetization of geometric theorems which are by no means self evident but which admit of an easy analytic proof.

We shall be concerned with a single closed curve, C, without double points and with continuously turning tangent.

Arithmetically. We shall be concerned with a pair of functions, $x(s)$, $y(s)$, such that,

(a) $x(s+nl)=x(s)$, $y(s+nl)=y(s)$, $x'(s+nl)=x'(s)$, $y'(s+nl)=y'(s)$, where l is constant and n is any integer;

(b) $x'(s)$ and $y'(s)$ are continuous;

(c) $x'^2(s)+y'^2(s)=1$;

(d) The equations $\left.\begin{matrix} x(t)=x(s) \\ y(t)=y(s) \end{matrix}\right\}$ have no simultaneous solution, (t, s), except $t=s+nl$.

From the above hypotheses follows the uniform continuity of $x(s)$, $y(s)$, $x'(s)$, $y'(s)$.

We pass now to the theorems, giving them first a geometric form, then changing this to the arithmetic form and supplying the proof.

Theorem I. *It is possible to describe about any point of the curve, C, a circle which shall cut from C at most one segment, namely, that passing through the center of the circle; moreover the same radius may be made to serve for the circle at all points of C.*

Let us write

$$\rho\,(t, s)=\sqrt{[x(t)-x(s)]^2+[y(t)-y(s)]^2},$$

where here and always in the following, the positive square root is meant.

The function $\rho\,(t, s)$ is continuous in both variables; it therefore attains its maximum, R, say, for $t=t_m$, $s=s_m$. Choose $0<r<R/2$. Then for fixed t, the equation $\rho\,(t, s)=r$ has a solution; for $\rho\,(t, t)=0$, and as

$$\rho\,(t, s_m)+\rho\,(t, t_m)>(t_m, s_m)=R,$$

either $\rho\,(t, s_m)$ or $\rho\,(t, t_m)$ is greater than $R/2$ and hence than r. Thus the

*See "Potential Functions on the Boundary of their Regions of Definition," *Transactions of the American Mathematical Society*, Vol. 9, No. 1, pp. 43 and 49.

continuous function $\rho\,(t, s)-r$, which is positive for $s=t$ and negative at either $s=t_m$ or $s=s_m$ must vanish at some intermediate point.

Calling $s-t=h$, the equation $\rho\,(t,\ t+h)=r$ has a first positive solution $h=P(t, r)$, and a first negative solution $h=N(t, r)$. Our theorem I may now be stated arithmetically:

It is possible to choose r, independent of t, so small that the inequality,

$$\rho\,(t, s)<r, \tag{1}$$

holds only for values of s and t satisfying the inequalities

$$\left.\begin{array}{l} N(t, r)<s-t+nl<P(t, r) \\ i.\ e.,\ N(t, r)<h+nl<P(t, r) \end{array}\right\} \tag{2}$$

Passing to the proof, we observe first of all that $\rho\,(t,\ N(t, r))=r$ and $(t,\ \rho(t,\ P(t, r)))=r$, and we shall be concerned with choosing r, independently of t so that, first, as $h+nl$ passes out from the interval $N(t, r)<h+nl<P(t, r)$, $\rho\,(t,\ t+h)$ becomes greater than r, and secondly, that it remains so.

As for the first, $\rho\,(t,\ t+h)$ for fixed t, increases with $|\,h\,|$. For

$$\frac{\frac{\partial}{\partial h}\rho^2(t,\ t+h)}{h}=\frac{x(t+h)-x(t)}{h}.x'(t+h)\ +\frac{y(t+h)-y(t)}{h}.y'(t+h)$$

$$=x'(t+I_1.h)x'(t+h)+y'(t+I_2.h)y'(t+h)\quad (0\leqq I_1\leqq 1,\ 0\leqq I_2\leqq 1).$$

This function approaches 1 as h approaches 0, and as $x'(s)$ and $y'(s)$ are uniformly continuous, we can find a positive quantity, p, such that the function is positive so soon as $|\,h\,|<p$, and this independently of t. For h so restricted we have

$$\frac{2\,\rho}{h}\frac{\partial\,\rho}{\partial\,h}>0,\ \text{whence}\ \frac{\partial\,\rho}{\partial\,h}\ \text{has the sign of}\ h,$$

so that $\rho(t,\ t+h)$ increases with $|\,h\,|$ for $|\,h\,|<p$, as stated. Let now r_1 be the minimum of the continuous function of t, $\rho\,(t,\ t+p)$, in the closed interval $0\leqq t\leqq 1$. This minimum is actually attained, and cannot therefore be 0 because of hypothesis (d). If now as a first restriction upon r we make it less than r_1, the inequality (1) cannot hold in either of the intervals

$$-p<h<N(t, r),\quad P(t, r)<h<p,$$

and hence in any of the intervals,

$$-p<h+nl<N(t, r), \quad P(t, r)<h+nl<p.$$

As for the second part of the proof, that the inequality (1) cannot hold outside the interval from $-p$ to p (except for values of h congruent to values within the interval, modulo 1), we shall find it convenient to restrict ourselves to a fundamental region of the periodic function $\rho\ (t, s)$, namely, $0 \leqq t \leqq 1$, $0 \leqq s \leqq 1$. Every point (t, s) not already considered, that is, not satisfying the inequalities

$$-p<s-t+nl<p$$

has a representative point (for which the function has the same value) in the subregion

$$p \leqq | s-t | \leqq 1-p, \quad 0 \leqq t \leqq 1, \quad 0 \leqq s \leqq 1.$$

But this is a closed region, and in it $\rho\ (t, s)$ cannot vanish, by hypothesis (d), the function therefore has a positive minimum r_2. If now r be also taken less than r_2, $\rho\ (t, s)>r_2>r$ for all points of the region considered.

It has thus been shown that if $r<r_1$, and $r<r_2$, condition (1) can only hold when t and s satisfy the inequalities (2), as was to be proved.

Theorem II. *If normals be drawn to the curve C at all points of an arc of length* $2\,\alpha$, *and if segments of length* β *be measured off upon these normals to either side of C, a circle can be drawn with center at the mid-point of the arc of C considered, such that the entire surface of the circle will be covered by the segments of the normals; moreover one choice can be made for the radius of the circle which will hold for all situations of the arc upon C.*

Arithmetically this may be stated: *Given* α *and* β, *we can find r, independent of t, and such that for any point* (ξ, η) *satisfying the condition*

$$\rho(\xi, \eta; t)=\sqrt{\{[\xi-x(t)]^2+[\eta-y(t)]^2\}}<r,$$

the equations

$$\xi=x(t+h)-\lambda\, y'(t+h), \tag{3}$$
$$\eta=y(t+h)-\lambda\, x'(t+h), \tag{4}$$

admit of at least one solution, (h, λ), *where* $| h | <\alpha$ and $| \lambda | <\beta$.

Evidently if proved for one α and β, the theorem holds for every greater pair of values. We shall therefore replace them by a single number, $2n$, smaller than either, and also less than p of the preceding paragraphs. Then r satisfies the requirements of the theorem if

$$r<r_3/3, \tag{5}$$

where r_3 is the minimum of $\rho(t, t+n)$. To prove it we consider the function

$$\rho^2(\xi,\ \eta;\ t+h)-r_3^2/4.$$

This is a continuous function of h, and for $h=0$ it is negative because of (3) and (5). On the other hand, it is positive for $h=n$ and also for $h=-n$, as we proceed to show. By the definition of r_3, $\rho(t,\ t+n)\geqq r_3$, and since

$$\rho(\xi,\ \eta;\ t)+\rho(\xi,\ \eta;\ t+n)>\rho(t,\ t+n)$$

we have

$$\rho(\xi,\ \eta;\ t+n)>r_3-\rho(\xi,\ \eta;\ t),$$

and as by conditions (3) and (5), $\rho(\xi,\ \eta;\ t)<r<r_3/3$, we have

$$\rho(\xi,\ \eta;\ t+n)>2r_3/3.$$

Hence

$$\rho^2(\xi,\ \eta;\ t+h)-r_3^2/4>0,$$

for $h=n$, and similarly for $h=-n$.

It follows that this function vanishes at some point $h=h'$ between 0 and n, and also at some point $h=h''$ between 0 and $-n$. But its derivative is also continuous and hence, by Rolle's theorem, vanishes at some point between $h=h'$ and $h=h''$, say $h=\bar{h}$:

$$[\xi-x(t+\bar{h})]x'(t+\bar{h})+[\eta-y(t+\bar{h})]y'(t+\bar{h})=0.$$

As $x'^2(t+\bar{h})+y'^2(t+\bar{h})=1$, both $x'(t+\bar{h})$ and $y'(t+\bar{h})$ do not vanish; to fix ideas, suppose the first is different from 0. Putting the ratio

$$\frac{\eta-y(t+\bar{h})}{x'(t+\bar{h})} \text{ equal to } \bar{\lambda},$$

we have

$$\eta=y(t+\bar{h})+\bar{\lambda}x'(t+\bar{h}),\quad \xi=x(t+\bar{h})-y'(t+\bar{h}),$$

showing that $\bar{h}$ and $\bar{\lambda}$ are a solution of the equations (4). It is at once clear that $\bar{h}$ obeys the restriction laid upon it. To see the same for $\bar{\lambda}$, transpose in the above equations the first terms on the right, then square and add. The result is

$$\bar{\lambda}=\pm\rho(\xi,\ \eta;\ t+\bar{h}).$$

But

$$\rho(\xi,\ \eta;\ t+\bar{h})\leqq\pi(\xi,\ \eta;\ t)+\rho(t,\ t+\bar{h}),$$

in which $\rho(\xi,\ \eta;\ t)<r<r_3/3$, and because $\rho(t,\ t+h)$ increases as $|\,h\,|$ increases, $\rho(t,\ t+\bar{h})<\rho(t,\ t+n)<r_3$,

$$|\bar{\lambda}| < 4r_3/3,$$

that is, since r_3 is the chord belonging to an arc of length n, $|\bar{\lambda}| < 4n/3$, and hence $<2n$, and the theorem is proved.

It does not of course follow from the above that through a given point of the surface of the circle *only one* normal passes. Indeed if part of the curve C be of the form $y=x^{\frac{4}{3}}$, which has infinite curvature at the origin, and yet has a continuously turning tangent, it is clear that there are points as near as we please to the curve through which there are three normals, all belonging to as short an arc of C as we please. A corollary to theorem I, however, supplies us with a uniqueness theorem which is often just as useful in the applications:*

The chords of the circle in theorem I which are perpendicular to the tangent at C at the center of the circle, cut C but once. In other words, if instead of taking a little field of normals about a point of C, we take a field of straight lines through the points of a limited arc of C and parallel to the normal to C at the mid-point of the arc, these lines will cover just once the surface of a little circle.

The equations of one of these parallels to the normal to C through the point $x(t)$, $y(t)$ will be

$$\xi=x(t)+\delta x'(t)-\mu y'(t),$$
$$\eta=y(t)+\delta y'(t)+\mu x'(t),$$

where δ is the distance of the parallel from $x(t)$, $y(t)$, and μ is a parameter. Arithmetically therefore, the theorem may be stated:

The equations

$$x(t+h)=x(t)+\delta x'(t)-\mu y'(t),$$
$$y(t+h)=y(t)+\delta y'(t)+\mu x'(t),$$

admit of only one solution for which $\rho(t, t+h)<r$.

To see this, eliminate μ. We find that we have merely to show that the function

$$F(h)=x'(t)[x(t+h)-x(t)]+y'(t)[y(t+h)-y(t)]-\delta,$$

has at most one solution for which $\rho(t, t+h)<r$. But

$$F'(h)=x'(t)x'(t+h)+y'(t)y'(t+h),$$

which we have previously seen to be positive for $|h|<p$, and this includes

*See Liapounoff, Sur certaines questions qui se rattachent au probleme de Dirichlet; Journal de Math. Liouville, 1898. The author makes an hypothesis upon his bounding surface which the above reasoning, carried out for space, would show to be unnecessary.

all values of h corresponding to $\rho(t, t+h)<r$, and hence $F(h)$, being an always increasing function in the interval, can vanish at most but once. We omit the simple proof that if $\delta<r$ and if r is small enough $F(h)$ *does* vanish once.

DIVISIONS OF AN ANGLE INTO EQUAL PARTS BY MEANS OF A TRANSCENDENTAL CURVE.

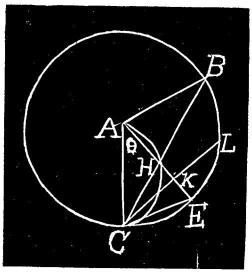
By J. S. BROWN, San Marcos, Texas.

The problem of trisecting an angle by means of the cardioid is capable of a much more general setting as follows:

Given CAE an isosceles triangle, $AC=AE$, angle $A=\theta$; also given a point H in AE such that angle $HCE=n.\theta$. Required, the locus of H if AC is fixed and AE rotates in the plane about A. E will describe a circle with A as center and AE as radius, as indicated in the figure.

Draw CL perpendicular to AE; angle $LCE=\frac{1}{2}\theta$; $CK=CH\cos(n-\frac{1}{2})\theta$, and $CK=AC\sin\theta$. Then

$$CH=\frac{AC\sin\theta}{\cos(n-\frac{1}{2})\theta}, \qquad (1)$$



which is the polar equation of the locus of H when C is taken as origin and AC is the axis of reference.

By assigning the successive values $\frac{1}{2}$, 1, $1\frac{1}{2}$, etc., to n the particular equations of the curves by means of which subdivisions one-half, one-third, etc., of an angle can be constructed, are found.

As an example, suppose it is desired to find the equation of the curve by means of which an angle with A as vertex and AC as one side, can be trisected; for example, angle BAC.

It is evident that in this case $n=1$. Substituting this value of n in the general equation, we find

$$CH=\frac{AC\sin\theta}{\cos\frac{1}{2}\theta}=\frac{AC.2\sin\frac{1}{2}\theta.\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}=AC.2\sin\frac{1}{2}\theta=2AC\sin\frac{1}{2}\theta. \qquad (2)$$

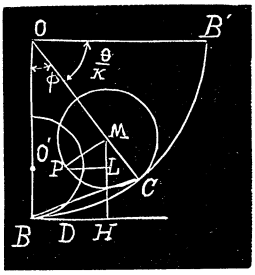
Suppose AHC to be the curve for this value of n, and let H be the point where BC cuts this curve. Draw AHE. The angle $CAE=\frac{1}{3}CAB$, for angle $BCE=\theta$ and angle $BAE=2\theta$.

Evidently, this curve corresponding to (2) is the cardioid, though the

form of the solution here used is quite different from the usual method of trisection by the cardioid.

In this manner the curve corresponding to any value of n as above indicated can be found. By means of an instrument constructed on the principle which governs the relative rates of motion of the hands of a clock, any of these curves can be accurately drawn.

The hypocycloid curve may also be used for the purpose of general equi-section of an angle, as appears in the following:



Given OB the radius of a circle, and suppose $O'B=r=\frac{OB}{k.n}$ where $\frac{\theta}{360}=\frac{1}{n}$, and k is the number of equal parts into which it is required to divide the angle θ.

B and B' are two consecutive points common to the hypocycloid and the circumference of the circle whose center is O, the rolling circle having the radius $O'B=r$. It is evident that the angle $BOB'=\theta/k$. To find the equation of the curve, let B be the origin, BH the X-axis, BO the Y-axis, ϕ the angle BOC, P a point on the curve, $x=BD$ and $y=PD$. Then $y=MF+FH-ML=r+FH-ML$.

But $ML=r\cos(kn-1)\phi$, $FH=(BD+DH)\tan\frac{1}{2}\phi=(x+DH)\tan\frac{1}{2}\phi$, $DH=PL=r\sin(kn-1)\phi$.

Hence, $y=r+[x+r\sin(kn-1)\phi]\tan\frac{1}{2}\phi-r\cos(kn-1)\phi$, the equation of the hypocycloid, in which it may be noted that angle $PMC=k.n.\phi$.

While the equation is general, the construction of the curve can be made only when the ratio of the given angle to the perigon is known.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

290. Proposed by G. I. HOPKINS, M. A., Manchester, New Hampshire.

A and B are 45 miles apart and travel towards each other. A goes one mile the first day, three miles the second day, five miles the third day, and so on. B goes two miles the first day, four the second, six the third, and so on. In how many days will they meet? What interpretation is to be placed upon the negative value of n?

Solution by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

Let n=the required number of days for A and B to meet. When a,

d, and n are given in an arithmetical progression to find s we have

$$s=n[2a+d(n-1)]/2. \tag{1}$$

When for A we have $a=1$, and $d=2$, we have from (1), $s=n[2+2(n-1)]/2=n^2=$the distance traveled by A. When for B we have $a=2$ and $d=2$, we have from (1), $s=n[4+2(n-1)]/2=(n^2+n)=$the distance traveled by B.

$$\therefore 2n^2+n=45;\text{ whence } n=4.5 \text{ or } -5. \tag{2}$$

When $n=4.5$, $n^2=20.25=$distance traveled by A. We then have $(n^2+n)=24.75=$the distance traveled by B. Or $20.25+24.75=45$.

When $n=-5$, $n^2=25=$distance traveled by A. We then have $(n^2+n)=(-5)^2+(-5)=25-5=20=$the distance traveled by B. Or $25+20=45$.

The positive value, 4.5, and the negative value, -5, may be interpreted as follows: Draw a circle 90 miles in circumference, and let A stand at N, the north pole, and B at S, the south pole, 45 miles from each other, measured on either semi-circumference. In the $4\frac{1}{2}$ days A will walk 20.25 miles from N along the eastern semi-circumference until he meets B who has walked 24.75 miles from S along the eastern semi-circumference. A's position is now southeast from N, 81°; and B's position is now northeast from S, 99°; or they are theoretically at the same point. In the -5 days, or 5 days ago, A and B were theoretically at the same point on the western circumference 100° from N or 80° from S, A walked towards N and traveled 25 miles, while B walked towards S and traveled 20 miles, then A is at N, and B at S.

NOTE. This problem was also solved by J. E. Sanders, who obtained as a result $4\frac{9}{19}$ days. Since at the end of the 4th day the two men are 9 miles apart, Mr. Sanders assumes that A, who would travel 10 miles the 5th day provided he kept up the law of travel of the four previous days, would, in the time, t, travel $10t$, and B on the same conditions would travel $9t$; and, therefore, both would travel $19t=9$ miles. Whence $t=\frac{9}{19}$. Hence the whole time is $4\frac{9}{19}$ days. Mr. Sanders interprets -5 days as the answer to the problem, "A and B are 45 miles apart and travel towards each other. A goes 1 mile the first day, 3 miles the second day, 5 miles the third, and so on. B goes 0 miles the first day, 2 miles the second day, 4 miles the third day, and so on." While Mr. Sanders' interpretation, as well as that of Mr. DeLand, is ingenious, and perhaps also correct, yet such interpretations seem to be *strained*. By the insertion of a few words, one would obtain the same set of results while the interpretations would fail to *interpret*.

Thus, suppose the men were restricted to walk on the straight line joining them. This would nullify Mr. DeLand's interpretation as to their traveling in a circle. In the case A and B are to travel in a straight line joining them, 4½ days as given by Mr. DeLand, is the most satisfactory, and he has furnished, in a letter to the editor, some very convincing arguments drawn from practical interpolation formulae to support his view. However, whatever interpretation lawyers and judges might put on similar problems involving similar principles when applied to government bonds, etc., yet the universal testimony of the best mathematicians would be to the effect that the problem is *indeterminate*, for the statement of the problem is silent as to the rates of travel of the two men on the respective days. Mr. Sanders assumes that, on the fifth day, they travel uniformly, while Mr. DeLand assumes that A travels 4.75 miles, and B 4.25 miles in the forenoon, and were they to keep up the law of travel as on previous days A would travel 5.25 miles and B 4.75 miles in the afternoon. Any other conceivable rate of travel each day might be assigned. The problem only requires them to travel a certain number of miles each successive day, but leaves the rates of travel to the ingenuity of the interpreter.

As to negative results, there is some diversity of opinion among good algebraists. Thus, for example, C. Smith, *Treatise on Algebra*, p. 259, says, "It should be remarked that a negative value of n cannot mean a number of terms reckoned *backwards*;" while Hall and Knight, *Higher Algebra*, page 33, in their solution of the prob-

lem, "How many terms of the series $-9, -6, -3, \ldots$, must be taken that the sum may be 66," say, "If we begin at the last of these terms and *count backwards* four terms, the sum is also 66."

Fine, *College Algebra*, page 355, gives the following example: "Given $d=\frac{1}{2}$, $l=\frac{3}{2}$, $s=-\frac{15}{2}$; find a and n." In reference to the two values of n, viz., 10 and -3, Professor Fine says, "The value of $n=-3$ is inadmissible." So say Fisher and Schwatt, *Higher Algebra*, page 382. But they say such results may be assumed to mean that the terms be counted backwards.

The weight of authority seems to be that in such problems negative and fractional values are uninterpretable.
ED. F.

GEOMETRY.

324. Proposed by FRANK LOXLEY GRIFFIN, S. M., Ph. D., Instructor in Mathematics, Williams College, Williamstown, Mass.

Find all plane curves such that the normal lengths intercepted by the co-ordinate axes are in a constant ratio for all points.

Solution by PROFESSOR C. E. WHITE, Nashville, Tenn., and the PROPOSER.

Let $f(x, y)=0$ be the equation of the required curves. Then the normal at the point (x, y) has the equations:

$$\frac{\xi-x}{\dfrac{\partial f}{\partial x}}=\frac{\eta-y}{\dfrac{\partial f}{\partial y}}=\frac{-\rho}{R},$$

where (ξ, η) is any point of the normal, ρ the distance from (x, y) to (ξ, η), and $R=\sqrt{\left(\dfrac{\partial f}{\partial x}\right)^2+\left(\dfrac{\partial f}{\partial y}\right)^2}$. The normal length intercepted by the axes, $x=0$, and $y=0$, are the values of ρ for $\xi=0$, and for $\eta=0$, respectively. Thus,

$$\rho_x=x.\frac{R}{\dfrac{\partial f}{\partial x}} \text{ and } \rho_y=y.\frac{R}{\dfrac{\partial f}{\partial y}}.$$

By hypothesis, $k_1\rho_x=k_2\rho_y$, or

$$\frac{k_1x}{\dfrac{\partial f}{\partial x}}=\frac{k_2y}{\dfrac{\partial f}{\partial y}}. \tag{1}$$

But, since $\dfrac{\partial f}{\partial x}.dx+\dfrac{\partial f}{\partial y}.dy=0$,

$$k_1xdx+k_2ydy=0, \tag{2}$$

whence, c being an arbitrary constant,

$$k_1x^2+k_2y^2=c. \tag{3}$$

The required curves are, then, the central conics:—ellipses if the normal meets the two axes on the same side of the curve; and hyperbolas, if on opposite sides of the curve. The given property might be taken to define the central conic sections.

Also solved by G. B. M. Zerr.

CALCULUS.

254. Proposed by H. S. PARDEE, Boston, Mass.

A wire is wound in the form of a helix. Assuming that sections of the wire perpendicular to the axis of the wire are circles, find the equation of a section of the wire perpendicular to the axis of the helix.

Solution by DR. E. SWIFT, Princeton, University.

The shape of the wire after bending may be changed or distorted. The simplest assumption is that after bending, a section perpendicular to the helix is still a circle, say, of radius R. In this case, the surface of the wire may be regarded as the envelope of a family of spheres, whose centers are on the wire and whose radii are all equal to R. We may write the equation of the helix as

$$\left.\begin{array}{l} x=a\cos t \\ y=a\sin t \\ z=ct \end{array}\right\}$$

where t is parameter, a and c constants depending on the shape and size of the helix. The family of spheres has then for its equation,

$$(x-a\cos t)^2+(y-a\sin t)^2+(z-ct)^2=R^2. \tag{1}$$

To find the envelope we differentiate (1) with respect to t and eliminate t from (1) and the resulting equation. To find the equation of the section by a plane perpendicular to the axis, here the XY-plane, we must set $z=0$. If we do this before eliminating t, we obtain the equation of the plane section in the parametric form

$$\left.\begin{array}{l} (x-a\cos t)^2+(y-a\sin t)^2+(-ct)^2=R^2 \\ a\sin t(x-a\cos t)-a\cos t(y-a\sin t)+c^2t=0 \end{array}\right\} \tag{2}$$

and these equations show that the section is *the envelope of circles whose centers lie on the circle* $\left.\begin{array}{l} x=a\cos t \\ y=a\sin t \end{array}\right\}$ *and whose radii are* $\sqrt{(R^2-c^2t^2)}$.

It is possible to solve these equations for x and y in terms of t. Expanding them, we have

$$\left.\begin{array}{l} x^2-2ax\cos t+y^2-2ay\sin t+a^2+c^2t^2-R^2=0 \\ ax\sin t-ay\cos t+c^2t=0 \end{array}\right\} \qquad (3)$$

or

$$\left.\begin{array}{l} (x^2+y^2)+a^2+c^2t^2-R^2=2a(x\cos t+y\sin t) \\ 2c^2t=2a(-x\sin t+y\cos t) \end{array}\right\}. \qquad (4)$$

Squaring and adding, we have

$$(x^2+y^2)^2+2(x^2+y^2)(a^2+c^2t^2-R^2)+(a^2+c^2t^2-R^2)^2+4c^4t^2=4a^2(x^2+y^2)$$

a quadratic equation for x^2+y^2, or the square of the radius vector r. And if we use polar coordinates, $\left.\begin{array}{l} x=r\cos\phi \\ y=r\sin\phi \end{array}\right\}$, this equation enables us to find r. Putting for x and y, $r\cos\phi$ and $r\sin\phi$, respectively, in the last of equations (4), we have

$$c^2t=ar\sin(\phi-t), \text{ or } \phi=t+\sin^{-1}\frac{c^2t}{ar}.$$

The equations of the curve are then

$$r=\sqrt{[2\sqrt{(a^2R^2-a^2c^2t^2-c^4t^2)}-(c^2t^2-a^2-R^2)]}, \quad \phi=t+\sin^{-1}\frac{c^2t}{ar}.$$

Also solved by G. B. M. Zerr. An incomplete solution was received from the Proposer.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

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145. Proposed by J. D. WILLIAMS, being the 12th of his fourteen challenge problems proposed in 1832.

Make $x^2+y^2=\square$, $\frac{5}{4}(x^2+y^2)=$a cube, $xy=2x^3$, $2(x+y)+\dfrac{xy}{x+y}=\square$, and $(x^4+y^4)(x^2+y^2)-(x^5+y^5)\sqrt{(x^2+y^2)}=\square$.

Solution by DR. E. SWIFT, Princeton University.

If $x^2+y^2=\square$, we must have $x=2\xi\eta k$, $y=(\xi^2-\eta^2)k$, where ξ, η, k are integers. Then $x^2+y^2=k^2[\xi^2+\eta^2]^2$,

If $\frac{5}{4}$ of this=a cube, evidently it must be of the form $4\times25\times$a cube, or $k(\xi^2+\eta^2)$ is of the form $2\times5\times$a cube.

Since $xy=2x^3$, either, a) $x=0$, or, b) $y=2x^2$.

If a) is true, $y=\sqrt{(x^2+y^2)}$ and must be of the form $2\times5\times$a cube.

$x=0$ satisfies the last condition; it remains to satisfy the third, which reduces to $2y=\square$

Since $y=10a^3$, the least value of a which makes $2y^2$ square is 5, and $y=10\times125=1250$.

The values $x=0$, $y=1250$ are the smallest values satisfying the conditions when $x=0$. The complete solution is $x=0$, $y=1250a^6$, a being any number.

If $x\neq 0$, $y=2x^2$, and we must have, putting in the values of x and y in terms of ξ and η,

$$(\xi^2-\eta^2)k=4\xi^2\eta^2k^2, \text{ or } \xi^2-\eta^2=4k\xi^2\eta^2,$$

a second relation between ξ and η. But this is evidently impossible, if ξ and η have integral values, as is necessary if x and y are to be integral; for transposing $\xi^2=\eta^2(4k\xi^2+1)>\xi^2$. The only *integral* solutions, then, are $x=0$, $y=1250a^6$.

147. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

If $4n+3$ is prime, $2(1.2.3...4n)+1\equiv 0 \pmod{4n+3}$; and conversely. If $4n+3$ is prime, $(1.2.3...2n)^2-4\equiv 0 \pmod{4n+3}$; and conversely.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

By Wilson's Theorem, $1+(4n+2)!\equiv 0(\text{mod. } 4n+3)$.

$\therefore 1+(4n+2)(4n+1)(4n)!\equiv 0(\text{mod. } 4n+3)$;

$1+[4n(4n+3)+2](4n)!\equiv 0(\text{mod. } 4n+3)$.

$\therefore 1+2(1.2.3.4...4n)\equiv 0(\text{mod. } 4n+3)$.

By a corollary to Wilson's Theorem, $[(2n+1)!]^2-1\equiv 0(\text{mod. } 4n+3)$.

$\therefore (2n+1)^2[(2n)!]^2-1\equiv 0(\text{mod. } 4n+3)$;

$[n(4n+3)+n+1][(2n)!]^2-1\equiv 0(\text{mod. } 4n+3)$.

$\therefore 4(n+1)[(2n)!]^2-4\equiv 0(\text{mod. } 4n+3)$.

$\therefore (4n+3)[(2n)!]^2+[(2n)!]^2-4\equiv 0(\text{mod. } 4n+3)$.

$\therefore (1.2.3.4...2n)^2-4\equiv 0(\text{mod. } 4n+3)$.

The converse follow at once.

AVERAGE AND PROBABILITY.

192. Proposed by PROFESSOR R. D. CARMICHAEL, Anniston, Ala.

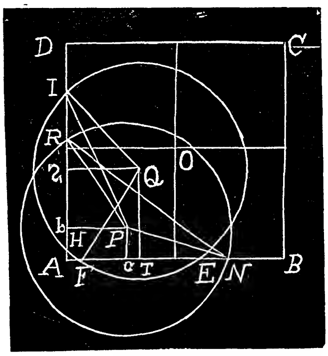
A point is taken at random in a square whose side is a point as center and radius$=a$ a circumference is described. W mean area of that part of the circle which lies within the square?

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

By considering the square AO, $\frac{1}{4}$ the original square AC, all possible positions of the circle are taken. Let P, Q be the center of the circle, P taken in the quadrant of a circle radius a, center A, and Q taken in AO

without this quadrant. Coordinates of P, $(Pa, Pb)=(x, y)$; coordinates of Q, $(QT, QS)=(u, v)$.

Then $G=\pi a^2-$area of segment on $IH-$area of segment on FE is the area of Q required.



$K=\frac{1}{2}AR\times AN+$area of segment on RN is the area of P required.

Let $\angle TQF=\theta$, $\angle IQS=\phi$, $\angle NPa=\psi$, $\angle RPb=\rho$.

$\therefore u=a\cos\theta$, $v=a\cos\phi$, $x=a\cos\psi$, $y=a\cos\rho$, $AN=a\cos\rho+a\sin\psi$, $AR=a\cos\psi+a\sin\rho$, $\angle RPN=3\pi/2-\psi-\rho$.

$\therefore G=(\pi-\theta-\phi+\sin\theta\cos\theta+\sin\phi\cos\phi)a^2$;

$K=\frac{1}{2}(3\pi/2-\psi-\rho+\cos\rho\sin\rho+\cos\psi\sin\psi+2\cos\psi\cos\rho)a^2$.

The limits of θ are $\frac{1}{2}\pi-\phi$ and 0; of ϕ, 0 and $\frac{1}{2}\pi$; of ψ, $\frac{1}{2}\pi-\rho$ and $\frac{1}{2}\pi$; of ρ, 0 and $\frac{1}{2}\pi$. Then Δ, the required average area, is

$$\Delta=\frac{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi-\phi}Ga^2\sin\phi\sin\theta\,d\phi\,d\theta+\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi-\rho}^{\frac{1}{2}\pi}Ka^2\sin\rho\sin\psi\,d\rho\,d\psi}{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi-\phi}a^2\sin\phi\sin\theta\,d\phi\,d\theta+\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi-\rho}^{\frac{1}{2}\pi}a^2\sin\rho\sin\psi\,d\rho\,d\psi}$$

$$=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi-\phi}G\sin\phi\sin\theta\,d\phi\,d\theta+\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi-\rho}^{\frac{1}{2}\pi}K\sin\rho\sin\psi\,d\rho\,d\psi$$

$$=\tfrac{1}{6}a^2\int_0^{\frac{1}{2}\pi}[6(\pi-\phi+\sin\phi\cos\phi)\sin\phi-3\sin^3\phi\cos\phi-2\cos^3\phi\sin\phi-2\sin\phi]d\phi$$

$=(\pi-\frac{29}{24})a^2$, ϕ and ρ having the same limits, $\phi=\rho$.

MISCELLANEOUS.

155 (Incorrectly numbered 151). Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Sum the series $\sum\limits_{r=1}^{r=m}\operatorname{cosec}\left(\frac{2r-1}{4m}\pi+\theta\right)\operatorname{cosec}\theta\left(\frac{2r-1}{4m}\pi-\theta\right)$.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

$$S=\sum_{r=1}^{r=m}\operatorname{cosec}\left(\frac{2r-1}{4m}\pi+\theta\right)\operatorname{cosec}\left(\frac{2r-1}{4m}\pi-\theta\right)$$

$$=\sum_{r=1}^{r=m}\frac{\sin\left[\frac{2r-1}{4m}\pi+\theta-\left(\frac{2r-1}{4m}\pi-\theta\right)\right]}{\sin2\theta\sin\left(\frac{2r-1}{4m}\pi+\theta\right)\sin\left(\frac{2r-1}{4m}\pi-\theta\right)}$$

$$=\operatorname{cosec}2\theta\sum_{r=1}^{r=m}\left[\cot\left(\frac{2r-1}{4m}\pi-\theta\right)-\cot\left(\frac{2r-1}{4m}\pi+\theta\right)\right]$$

$$=\operatorname{cosec}2\theta\sum_{r=1}^{r=m}\left[\cot\left(\frac{2r-1}{2m}\pi-2\theta\right)-\cot\left(\frac{2r-1}{2m}\pi+2\theta\right)\right.$$

$$\left.+\operatorname{cosec}\left(\frac{2r-1}{2m}\pi-2\theta\right)-\operatorname{cosec}\left(\frac{2r-1}{2m}\pi+2\theta\right)\right].$$

But $\operatorname{cosec}\left(\frac{2r-1}{2m}\pi\mp2\theta\right)=\operatorname{cosec}\left(\frac{2m-2r+1}{2m}\pi\pm2\theta\right).$

$$\therefore\sum_{r=1}^{r=m}\left[\operatorname{cosec}\left(\frac{2r-1}{2m}\pi-2\theta\right)-\operatorname{cosec}\left(\frac{2r-1}{2m}\pi+2\theta\right)\right]=0.$$

$$\therefore S=\operatorname{cosec}2\theta\sum_{r=1}^{r=m}\left[\cot\left(\frac{2r-1}{2m}\pi-2\theta\right)-\cot\left(\frac{2r-1}{2m}\pi+2\theta\right)\right].$$

From trigonometry, $\prod_{r=0}^{r=m-1}\sin\left(\frac{r\pi}{m}+\phi\right)=\frac{\sin m\phi}{2^{m-1}}.$

Differentiating both sides, and reducing, we get,

$$\sum_{r=0}^{r=m}\cot\left(\frac{r\pi}{m}+\phi\right)=m\cot m\phi.$$

Let $\phi=\frac{\pi}{2m}-2\theta$, and also $\frac{\pi}{2m}+2\theta$. Then we get,

$$S=\operatorname{cosec}2\theta\sum_{r=1}^{r=m}\left[\cot\left(\frac{2r-1}{2m}-2\theta\right)-\cot\left(\frac{2r-1}{2m}\pi+2\theta\right)\right]$$

$$=m\operatorname{cosec}2\theta\left[\cot m\left(\frac{\pi}{2m}-2\theta\right)-\cot m\left(\frac{\pi}{2m}+2\theta\right)\right]=2m\operatorname{cosec}2\theta\tan2m\theta.$$

Also, $\sum_{r=1}^{r=m}\sec\left(\frac{2r-1}{4m}\pi+\theta\right)\sec\left(\frac{2r-1}{4m}\pi-\theta\right)=2m\operatorname{cosec}2\theta\tan2m\theta=S$ above.

This follows because

$$\sec\left(\frac{2r-1}{4m}\pi\pm\theta\right)=\operatorname{cosec}\left(\frac{2m-2r+1}{4m}\pi\mp\theta\right).$$

PROBLEMS FOR SOLUTION.

GEOMETRY.

333. Proposed by J. B. MORRELL, Boulder, Colorado.

Exhibit the fallacious argument to prove that a right-angle is equal to an angle which is less than a right-angle.

334. Proposed by J. O. MAHONEY, B. E., M. Sc., Central High School, Dallas, Texas.

Through any point P in the plane of the triangle ABC, draw a line that shall divide the perimeter of the triangle into two equal parts.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

152. Proposed by H. S. VANDIVER, Bala, Pa.

Prove geometrically:

$\sum_{n=1}^{\frac{1}{2}(p-1)}\left[\frac{n^2}{p}\right]=\frac{p-3}{4}\cdot\frac{p-1}{2}-\sum_{n=1}^{\frac{1}{4}(p-4}\left[\sqrt{np}\right]$, where $p\equiv3(\text{mod. }4)$ and $\left[\frac{k}{p}\right]$ represents the greatest integer in k/p.

AVERAGE AND PROBABILITY.

196. Proposed by R. D. CARMICHAEL, Anniston, Ala.

A circle is inscribed in a square. Find the chance that the distance between two points within the square and without the circle shall not exceed a side of the square.

197. Proposed by HENRY HEATON, Belfield, N. D.

Solve No. 188 on the supposition that all lines having the same direction are equally distributed in space, and lines passing through the same point are distributed as the radii of a sphere drawn to points equally distributed.

MISCELLANEOUS.

177. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Sum the infinite series:

(a) $\sin x+nx\cos x-\frac{n^2x^2}{2!}\sin x-\frac{n^3x^3}{3!}\cos x+\frac{n^4x^4}{4!}\sin x+\ldots,$

(b) $\cos x-nx\sin x-\frac{n^2x^2}{2!}\cos x+\frac{n^3x^3}{3!}\sin x+\frac{n^4x^4}{4!}\cos x\ldots$

NOTES AND NEWS.

The next summer meeting of the American Mathematical Society will be held at the University of Illinois, September 10-11, 1908.

Professor Oscar Bolza, of the University of Chicago, leaves at the close of the present session for a year's leave of absence in Europe.

Dr. J. H. McDonald, of the University of California, has been advanced to an assistant professorship in Mathematics for the coming year.

Associate Professor Kurt Laves of the Department of Astronomy, University of Chicago, is on leave of absence for a year from last April and is spending his vacation in Berlin.

Dr. N. J. Lennes, of the Massachusetts Institute of Technology, has been appointed to give instruction in Mathematics in the summer school at Chatauqua, New York, July 5th to August 14th.

At the summer session, 1908, of Columbia University, courses will be offered in Solid Geometry, Trigonometry, College Algebra, Analytic Geometry, Differential and Integral Calculus, Modern Higher Algebra, Differential Equations, Advanced Calculus, and Theory of Functions of a Complex Variable.

At the thirty-seventh annual session of the Kentucky Educational Association, to be held at Frankfort, Kentucky, June 16 to 18, 1908, Professor A. L. Rhoton, of Georgetown College, will be chairman of the mathematical section, and papers will be read on the Teaching of High School Mathematics by numerous representative teachers of the state.

BOOKS.

Elementary Algebra for Secondary Schools. By J. W. A. Young, Ph. D., Associate Professor of the Pedagogy of Mathematics, University of Chicago, and Lambert L. Jackson, Ph. D., formerly Professor of Mathematics, State Normal School, Brockport, N. Y. 12mo. Half leather. ix+438 pages. Price, $1.12. New York and Chicago: D. Appleton & Co.

This book is one of the Twentieth Century Series and is quite modern in the presentation of the subject matter of Algebra. In this treatment of Algebra, the authors kept in mind the logical value as well as the practical utility of the subject. The practical utility of algebra has been emphasized by introducing physical formulas, and by applying algebra to modern industrial, commercial, and scientific problems, the contents of which can be easily understood by the student. An unusual number of diagrams and illustrations are used throughout the book. The problems are numerous and well selected and cover nearly every phase of practical life.

B. F. F.

A Vest-Pocket Handbook of Mathematics for Engineers. By L. A. Waterbury, C. E., Professor of Civil Engineering, University of Arizona. First Edition; First Thousand. Morocco, vi+91 pages. Price, $1.00. New York: John Wiley & Sons.

This little handbook is intended as a ready reference for those who have studied or are studying the usual mathematics in engineering courses. It contains the fundamental formulas of Algebra, Trigonometry, Analytical Geometry, and the Calculus. It contains a number of important formulas on the mechanics of materials and several valuable tables. The book will be found quite convenient and useful for engineers and others who make use of mathematics in practical ways.

B. F. F.

College Algebra. By William H. Metzler, Ph. D., and Edward D. Roe, Jr., Ph. D., Professors of Mathematics in Syracuse University, and Warren G. Bullard, Ph. D., Associate Professor of Mathematics, Syracuse University. 8vo. Cloth. xiii+340 pages. Price, $1.50. New York: Longmans, Green & Co.

Some of the characteristic features claimed for this book are, conservative use of graphic representation of problems in physics, thoroughness, a combination of vigor and pedagogy, a full list of problems, originality in the treatment of certain subjects, and an endeavor to train the mind of the student.

In addition to the needs of the more advanced students, the ground required for entrance to scientific courses of the leading colleges and schools is quite thoroughly covered.

B. F. F.

Introduction to Infinitesimal Analysis. Functions of One Real Variable. By Oswald Veblen, Preceptor in Mathematics, Princeton University, and N. J. Lennes, Instructor in Mathematics in the Wendell Phillips High School, Chicago. First edition. First thousand. 8vo. Cloth. vii+227 pages. Price, $2.00. New York: John Wiley & Sons.

This volume is designed to serve not only as a convenient reference book in courses dealing with the fundamental theorems of the Infinitesimal Calculus, but also as a basis for a short course on real functions. By systematic use of the Heine-Borel theorem the authors have practically avoided the sequential division or "pinching" process commonly used in similar discussions. The authors aimed and, we believe secured, rigor of logic with the least amount of elaborate machinery. The work embraces nine chapters, of which the first deals with The Systems of Real Numbers; the second, with Sets of Points and of Segments; the third, with Functions in General—Special Cases of Functions; the fourth, Theory of Limits; the fifth, Continuous Functions; the sixth, Infinitesimals and Infinites; the seventh, Derivatives and Differentials; the eighth, Definite Integrals; ninth, Improper Definite Integrals.

B. F. F.

Elementary Algebra. By Frederick H. Somerville, B. S., The William Penn Charter School, Philadelphia. 8vo. Half leather. 407 pages. New York and Chicago. The American Book Co.

This book does not differ very essentially from the general standard of texts in Algebra. In the earlier chapters, exercises for oral drill are frequent. The introduction of some physical formulas will familiarize the student with the practical application of Algebra.

B. F. F.

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GENERALIZATION OF POSITIVE AND NEGATIVE NUMBERS.

By DR. G. A. MILLER, University of Illinois.

In the discussion of ordinary complex numbers it is generally made clear that the real numbers constitute a very special class of complex numbers, but the fact that this class is composed of all the complex numbers which are either positive or negative does not appear to have received sufficient emphasis. Calling α the amplitude or argument of a complex number we may group together all the numbers for which α has the same numerical value (that is, all the numbers situated on the same ray of the plane pencil whose vertex is the origin) and call them the α numbers. When $\alpha \doteq 0$, we obtain the positive numbers, and when $\alpha = \pi$, the negative ones. Hence the question arises whether we should not use the terms *zero-numbers* and *π-numbers* in place of positive numbers and negative numbers, respectively. Regardless of whether such a change of terms would be desirable it may be profitable to consider the positive and negative numbers from this standpoint, and this is the main object of the present note.

If α_0 represents any particular value of α it is clear that the sum of two α_0-numbers is necessarily an α_0-number; that is, all the numbers of the same ray have the group property* with respect to addition, but they clearly do not form a group with respect to this operation. The totality of numbers constituting the two rays α_0 and $\alpha_0+\pi$ evidently form a group with respect to either of the two operations, addition and subtraction, while the totality obtained by multiplying any one of them by all the real integers forms a subgroup with respect to either of these operations. With respect to the operations of addition and subtraction the two rays of real numbers do not present any group properties which differ from those of any other two rays which together form a straight line, but this is not the case with respect to the operations of multiplication and division as will appear more clearly from the following considerations.

It should be observed that the term division is here used with its most common meaning; viz., as the inverse of multiplication. Since a group in-

*Cf. Bocher, *Introduction to Higher Algebra,* 1907, p. 82.

volves the inverse of each of its operations it results that if a system of numbers forms a group when combined with respect to one of the fundamental operations of arithmetic it must also form a group with respect to the inverse of this operation. That is, if a system of numbers forms a group with respect to addition it also forms a group with respect to subtraction, and if it forms a group with respect to multiplication it also forms one with respect to division, and *vice versa*. The 0-numbers constitute the only ray of numbers which form a group with respect to multiplication. In fact, no other ray of numbers has even the group property with respect to either of the operations of multiplication and division, and if any finite number of rays have this property these rays must include the 0-numbers.

If any finite number of rays have the group property with respect to multiplication or division they evidently form a group with respect to multiplication but this is not necessarily true of special sets of numbers on such rays. For instance, the positive integers have the group property with respect to multiplication but they do not form a group with respect to this operation.* From the fact that the modulus of the product of two numbers is the product of the moduli of the factors it results that a system of numbers which form a group with respect to multiplication either includes only one number from a single ray or it includes an infinite number of numbers from every ray represented in the system. If the number of these numbers is finite they constitute the roots of an equation of the form $x^n=1$, and form a cyclic group since such an equation has primitive roots.

From the preceding paragraph it results that the rays as units form a cyclic group whenever numbers from a finite number of different rays are the elements of a group with respect to multiplication. *The ray of positive numbers is found in every such cyclic group,* while the ray of negative numbers occurs only in those of even order. Hence the positive numbers play a unique rôle in the groups of multiplication. While the ray of negative or π-numbers plays a less prominent rôle yet it enters into a larger number of the given finite groups of rays than any other ray except the one of positive numbers. A characteristic property of the positive and negative numbers is that they constitute the only two rays which form a group with respect to multiplication. It thus appears that while these two rays do not present any properties which differ from other straight lines of numbers with respect to the groups of addition they occupy a unique position in regard to the groups of multiplication; but their very special importance is due to the fact that they form a group with respect to both of the operations of addition and multiplication while no other finite number of rays has this property. This follows from the given results and the evident fact that *the pairs of rays which form a straight line constitute the only finite sets of rays of numbers which form a group with respect to addition.*

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*Weber, *Lehrbuch der Algebra*, Vol. 2, 1899, p. 4. It is to be noted that the term ray is used to represent the simply infinite system of numbers represented by the points on such a ray.

The preceding considerations lead to some direct extensions of the most fundamental rules of operating with positive and negative numbers. For example, the rule for the sums of two real numbers with opposite signs is included in the following: To obtain the sum of a number on a given ray and a number on the extended ray* find the difference of their absolute values and prefix to this the angle of that one whose absolute value is the greater. The fact that the sign of the product of a positive and a negative number is negative is included in the statement that the angle of the product of a number on the α_0 ray and a number on its extension is $2\alpha_0+\pi$. From this it results that the necessary and sufficient condition that a number is real is that a negative number is obtained by multiplying it by a number on the extension of its ray, and the necessary and sufficient condition that a number is a pure imaginary is that a positive number is obtained when we multiply the number into any number on the extended ray.

In the light of these results the question whether the terms positive and negative, which do not exhibit any evidence of the fact that they represent two special values of the possible amplitudes of a number, should be replaced, at least in theoretic work, by terms which exhibit their places in the infinite series of amplitudes, assumes a deeper meaning. If one should be asked to defend the terms 0-numbers and π-numbers it would be merely necessary to reply that the adjectives express the values of the amplitudes of these numbers, but no such rational defense could be made for the terms positive numbers and negative numbers. At any rate this view point seems to deserve notice since the question involved is so fundamental, and the present note makes no other claims for usefulness or novelty than the presentation of a very elementary and important matter in a somewhat new light.

ON CERTAIN PROPERTIES OF THE ORBITS OF A PARTICLE SUBJECT TO A CENTRAL FORCE VARYING AS AN INTEGRAL POWER OF THE DISTANCE.†

By E. J. MOULTON and F. H. HODGE, The University of Chicago.

In this paper are discussed the orbits of a particle moving subject to a central force varying as an integral power of the distance. The discussion is made with regard to their concavity and convexity, the number and distances of their apses, the ranges of values through which their radial dis-

*If the angles of two rays differ by π each one of them is said to be the extension of the other.

†This discussion had its origin in an exercise given by Professor F. R. Moulton to his class in Celestial Mechanics. It was read before the Chicago Section of the American Mathematical Society, April 18, 1908.

tances may vary, the amplitudes of the angles described by the radii vectores, and the time required for the maximum variation of the radial distance of the particle from the center of force. The discussion will be made from the implicit relations among the variables obtained from the first integrals.

It is shown in works on mechanics that the differential equations of the motion of a particle subject to a central force are

$$(1)\qquad \begin{cases} f=h^2u^2\left(n+\dfrac{d^2u}{d\theta^2}\right), & (n\geqq 0), \\ \dfrac{d\theta}{dt}=hu^2, \end{cases}$$

where f is the force, h is the constant of areas, u is the reciprocal of the radial distance to the particle, t is the time, and θ is the angle between the radius vector and some arbitrarily chosen reference line, the origin being at the center of force. If the force varies as an integral power of the distance we have

$$(2)\qquad f=\pm m^2h^2u^a,$$

where m^2h^2 is an arbitrary constant not equal to zero, and a is an integer. The positive sign signifies that it is an attractive force; the negative sign a repulsive force. From (1) and (2) we obtain

$$(3)\qquad \begin{cases} u+\dfrac{d^2u}{d\theta^2}=\pm m^2u^{a-2} & (u\geqq 0), \\ \dfrac{d\theta}{dt}=hu^2, \end{cases}$$

as the differential equations of motion.

Multiplying both sides of the first equation of (3) by $2\dfrac{du}{d\theta}$, and integrating, it is found that

$$(4)\qquad \begin{cases} u^2+\left(\dfrac{du}{d\theta}\right)^2=\pm\dfrac{2m^2}{a-1}u^{a-1}+c, & (a\neq 1,\ u\geqq 0), \\ u^2+\left(\dfrac{du}{d\theta}\right)^2=\pm 2m^2\log u+c, & (a=1,\ u\geqq 0), \end{cases}$$

where c is a constant of integration which is determined by the initial conditions. We then have

$$(5)\qquad \begin{cases} \theta = \displaystyle\int_{u_0}^{u} \frac{du}{\sqrt{\pm \dfrac{2m^2}{\alpha-1}u^{\alpha-1} - u^2 + c}}, & (\alpha \neq 1), \\[2ex] \theta = \displaystyle\int_{u_0}^{u} \frac{du}{\sqrt{\pm 2m^2 \log u - u^2 + c}}, & (\alpha = 1). \end{cases}$$

It is seen that when $\alpha = -1$, $+2$, 3, 4, or 5, u is a trigonometric, exponential, or elliptic function of θ, and the characteristics of the orbits are readily obtained. But when α is any other integer, u is in general not expressible by means of functions of θ which give readily the characteristics of the orbits.

It will be shown that by considering equations (3) and (4) the characteristic properties of the orbits for all integral values of α can be readily determined.

It is seen from physical considerations that for attractive forces the orbits are always concave toward the origin, and for repulsive forces always convex. This is evident also from equations (3) since a curve is concave toward the origin if $u + \frac{d^2u}{d\theta^2} > 0$, and convex if $u + \frac{d^2u}{d\theta^2} < 0$.

It is also to be noticed that when u changes from positive to negative the left member of the first equation of (3) changes sign, and that the right member changes sign if α is odd, but retains its sign if α is even. Hence, for $+m^2$ a negative value of u is allowable for attraction if α is odd, but must be considered as indicating repulsion if α is even. For $-m^2$ a corresponding statement is true. Remembering this, we can write (4) as follows:

$$(6)\qquad \begin{cases} \left(\dfrac{du}{d\theta}\right)^2 = \pm \dfrac{2m^2}{\alpha-1}u^{\alpha-1} - u^2 + c \equiv \phi(u), & (\alpha \neq 1), \\[2ex] \left(\dfrac{du}{d\theta}\right)^2 = \pm 2m^2 \log u - u^2 + c \equiv \phi(u), & (\alpha = 1). \end{cases}$$

We get from (3) and (6),

$$(7)\qquad dt = \frac{d\theta}{hu^2} = \frac{du}{hu^2\sqrt{\phi(u)}}.$$

Now $\frac{du}{d\theta} = 0$ is a necessary and sufficient condition for an apse since, if $\frac{du}{d\theta} = 0$, then $\frac{dr}{d\theta} = 0$, $(r = \frac{1}{u})$; that is, the orbit is perpendicular to the radius vector. And since $\left(\frac{du}{d\theta}\right)^2 \equiv \phi(u)$ the number and values of the roots of $\phi(u) = 0$ determine the number and distances of the apses of the orbits.

We can then discuss the orbits from the consideration of the graphs of $\phi(u)$. This is easily obtained by adding the corresponding ordinates of the graphs whose equations are, for $a \neq 1$,

$$y_1 = \pm \frac{2m^2}{a-1} u^{a-1}, \quad y_2 = -u^2, \quad y_3 = c;$$

and for $a=1$,

$$y_1 = \pm 2m^2 \log u, \quad y_2 = -u^2, \quad y_3 = c.$$

It appears in the construction of the graphs that the type of curve is different for different values of a, and that it is necessary to consider the following cases: I, $a>3$ and odd; II, $a>3$ and even; III, $a=3$; IV, $a=2$; V, $a=1$; VI, $a<1$ and even; and VII, $a<1$ and odd. It is also found that, except for $a=3$, we get the same types of curves, having the same number of maxima and minima, and giving the same combinations of real positive, real negative, zero, and multiple roots of $\phi(u)=0$, by taking a special value of the parameter m^2 and varying the parameter c, as by varying both m^2 and c. There is no loss in the generality of the results if we consider only a single value of m^2, with the exceptional case $a=3$, where the type of curve depends upon whether m^2 is greater than, equal to, or less than, unity.

The discussion is divided into seven cases, depending upon the value of a, and these are subdivided according to the value of c. The first case is given somewhat in detail, but in the other cases only the special peculiarities are pointed out. The time required for the maximum variation of u (which is denoted by T) is discussed after the other characteristics have been ascertained. Let Θ denote the angle described by the radius vector for the maximum variation of u.

CASE I. $a>3$ AND ODD.

The types of graph of $\phi(u)$ for the different values of c are shown in Fig. I, the sub-cases being (a), (b), (c), (d), and (e) for attraction, and (f), (g), and (h) for repulsion. Since a is odd, both positive and negative u's may be used.

Sub-Case (a). The characteristics of the orbits in this case are, (1) they have no apse, (2) they extend from the origin to infinity, and (3) Θ is finite.

It is evident that, since $\phi(u)=0$ has no real roots (as is seen from the graph), there can be no apse. If the initial conditions are $\theta=0$, $u=u_0$, $\frac{du}{d\theta}=\frac{du_0}{d\theta}$, then, since $\left(\frac{du}{d\theta}\right)^2>0$ for every u, $\frac{du_0}{d\theta}$ may be either positive or negative, but cannot be zero. If $\frac{du_0}{d\theta}>0$, then as θ increases u increases

until $\frac{du}{d\theta}$ becomes negative, which can occur only by $\left(\frac{du}{d\theta}\right)$ passing through

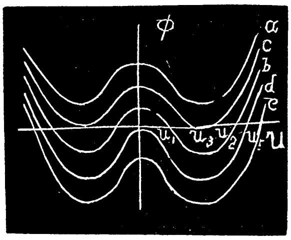


Fig. I. $a>3$ and odd.
Attraction.

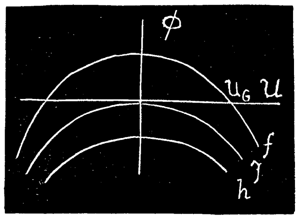


Fig. I. $a>3$ and even.
Repulsion.

infinity. Since $\left(\frac{du}{d\theta}\right)^2$ becomes infinite only for u infinite, u increases indefinitely. If $\frac{du_0}{d\theta}<0$, it follows from a similar argument that u decreases to negative infinity. Hence u may vary from $+\infty$ to $-\infty$. That is, $r=1/u$ may vary from 0 to $+\infty$, change to $-\infty$ and return to 0; or the orbit may extend from the origin to infinity and back to the origin.

The proof that Θ is finite comes from equations (5). Since the denominator of the integrand is never zero, and since $\sqrt{(\phi(\infty)}$ is infinite of order greater than unity, $\Theta=\int_0^\infty \frac{du}{\sqrt{\phi(u)}}$ is finite. This means that the orbit makes a finite number of revolutions around the origin between zero and infinity. The type of orbits for this case is shown in Fig. A.

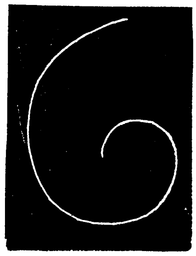


Fig. A.

Sub-Case (*b*). The characteristics of the orbits are (1) they have two apse distances, a single orbit, however, having only one apse distance, (2) the orbits lie either entirely within one circle, or entirely without a larger circle with the same center, the first orbit going to the origin, the second to infinity, and (3) Θ is finite for each orbit.

The real roots of $\phi(u)=0$ are $\pm u_1$, and $\pm u_2$; hence there are apses at $r=1/u_1$ and $r=1/u_2$. But a single orbit can have only one of these apse distances, since $\theta=\int_{u_1}^{u_2} \frac{du}{\sqrt{\phi(u)}}$ is imaginary, giving an imaginary orbit, between these two apse distances. The discussion of the orbits in this case is given in three parts:

(1) If the initial conditions are $\theta=0$, $u=u_0$, $0<|u_0|u_1$, $\frac{du}{d\theta}=\frac{du_0}{d\theta}$; then, if $\frac{du_0}{d\theta}>0$, u increases and, as may be seen from the graph, $\frac{du}{d\theta}$ decreases, becoming zero at $u=u_1$. Now $\frac{d^2u}{d\theta}=\frac{1}{2}\frac{d\phi(u)}{du}$. Therefore, as

is seen from the graph, $\frac{d^2u}{d\theta^2}\neq 0$ at $u=u_1$. Hence $\frac{du}{d\theta}$ continues to decrease and changes sign at $u=u_1$, and u begins to decrease. $\frac{du}{d\theta}$ does not change sign again until $u=-u_1$, where it becomes positive and u again increases to $u=+u_1$. This varying of u between $-u_1$ and $+u_1$ continues indefinitely as θ increases. If $\frac{du_0}{d\theta}<0$, the same range of variation of u results. We have then for the variation of r, $-u_1\leqq\frac{1}{r}\leqq+u_1$ or $\frac{1}{u_1}\leqq|\,r\,|\leqq\infty$. That is, the orbits lie outside the circle whose equation is $r=1/u_1$.

(2) If initially, $u=u_0$, $u_1<|\,u_0\,|<u_2$, then $\theta=\int_{u_0}^{u'_0}\frac{du}{\sqrt{\phi(u)}}$, $(u_1<|\,u'_0\,|<u_2)$, is imaginary. This means that there is no real orbit for these initial conditions and the given value of c.

(3) If the initial conditions are $\theta=0$, $u=u_0$, $u_2<u_0$, $\frac{du}{d\theta}=\frac{du_0}{d\theta}$, then, if $\frac{du_0}{d\theta}>0$, u increases indefinitely. If $\frac{du_0}{d\theta}<0$, u decreases to u_2 where $\frac{du}{d\theta}$ becomes zero and changes sign, after which u increases indefinitely. If $u_0<-u_2$ then $|\,u\,|$ has the same range of variation as if $u_0>u_2$; namely, $u_2\leqq|\,u\,|\leqq\infty$. Hence we have $0\leqq|\,r\,|\leqq 1/u_2$; that is, the orbits lie inside the circle whose equation is $r=1/u_2$, and have apses on this circle.

Since $\sqrt{[\phi(u)]}$ is an infinitesimal of order less than unity at $u=u_1$, $\Theta=\int_0^{u_1}\frac{du}{\sqrt{\phi(u)}}$ is finite. $\sqrt{[\phi(u)]}$ is also an infinitesimal of order less than unity at $u=u_2$ and is infinite of order greater than unity at $u=\infty$; therefore $\Theta=\int_{u_2}^{\infty}\frac{du}{\sqrt{\phi(n)}}$ is finite.

The type of orbits for this case is shown in Fig. B.

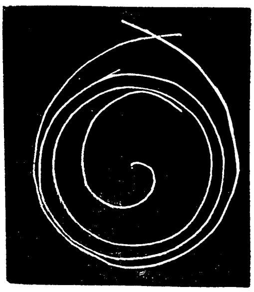


Fig. B.

Sub-Case (*c*). The characteristics of the orbits are (1) they have one apse distance, (2) they either extend from the origin to a certain circle whose center is at the origin, or always move on that circle, or extend from that circle to infinity, and (3), Θ is infinite for those orbits, which are not circles, the orbits approaching the circles asymptotically both from the inside and from the outside.

The real roots of $\phi(u)=0$ are $\pm u_3$; hence there is an apse at $r=1/u_3$. The discussion of the orbits is again broken up into three parts.

(1) If the initial conditions are $\theta=0$, $u=u_0$, $0\leqq|\,u_0\,|<u_3$,

$\frac{du}{d\theta}=\frac{du_0}{d\theta}$, then, if $\frac{du_0}{d\theta}>0$, u increases to $+u_3$; and if $\frac{du_0}{d\theta}<0$, u decreases to $-u_3$. Since $\phi(u)=0$ has a double root at $u=\pm u_3$, $\frac{1}{\sqrt{\phi(u)}}$ is infinite of the first order at $\pm u_3$. Hence $\Theta_1=\int_{u_0}^{\pm u_3}\frac{du}{\sqrt{\phi(u)}}$ is infinite. That is, the orbit approaches the circle $r=1/u_3$ asymptotically from the outside. Since $\frac{1}{\sqrt{\phi(u)}}$ is finite except at $u=\pm u_3$, the angle described by the radius vector going along the orbit from any point outside the circle to infinity is finite.

(2) If the initial conditions are $\theta=0$, $u=u_0=\pm u_3$, then $\frac{du_0}{d\theta}=0$, and $\frac{d^2u_0}{d\theta^2}=0$. Since $\theta=\int_{u_3}^{u'}\frac{du}{\sqrt{\phi(u)}}$ is infinite for any u' different from u_3, the orbit is the circle $r=1/u_3$.

(3) If the initial conditions are $\theta=0$, $u=u_0$, $u_0>u_3$, $\frac{du}{d\theta}=\frac{du_0}{d\theta}$, then if $\frac{du_0}{d\theta}>0$, u increases indefinitely as θ increases, and $\theta_1=\int_{u_0}^{\infty}\frac{du}{\sqrt{\phi(u)}}$ is finite. If $\frac{du_0}{d\theta}<0$, u decreases to u_3, and $\theta_2=\int_{u_0}^{u_3}\frac{du}{\sqrt{\phi(u)}}$ is infinite. That is, r decreases to zero from any point inside the circle $r=1/u_3$ for a finite value of θ, and increases to $1/u_3$ for an infinite value of θ. The orbit approaches asymptotically the circle $r=1/u_3$. The same orbit is obtained for $u_0<-u_3$.

Sub-Case (c) may be considered as the limiting case of *sub-case* (b), as the two circles approach coincidence: From this point of view one would say that the orbits have apses at $r=1/u_3$. It is also a limiting case of *sub-case* (a); whence the orbits might be considered as extending from the origin to infinity.

The type of orbits for this case is shown in Fig. C.

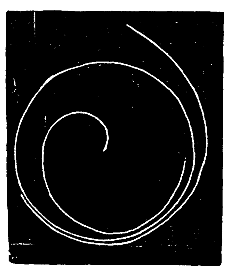


Fig. C.

Sub-Case (d). The characteristics of the orbits are, (1) they are either circles of infinite radius or finite orbits having one apse distance, (2) the finite orbits go to the origin, and (3) Θ is finite. The discussion in this case is essentially the same as that in *sub-case* (b). The outer orbit has become a circle at infinity.

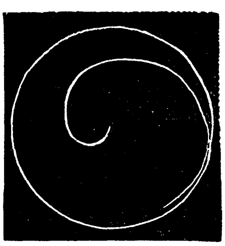


Fig. D.

Sub-Case (e). This is the same as the preceding sub-case except that the orbits at infinity have vanished.

Sub-Case (f). Here we are dealing with a repulsive force. The characteristics of the orbits are,

(1) they have one apse distance, and (2) they go to infinity, having a finite Θ. The discussion is made in the same manner as previously. It is seen that, since the orbit is convex toward the origin, $\Theta<\pi$.

Sub-Case (g). The orbit has gone to infinity.

Sub-Case (h). No real orbits exist.

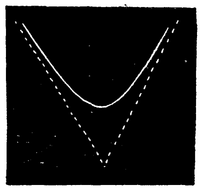


Fig. E.

CASE II. $\alpha>3$ AND EVEN.

The types of graph of $\phi(u)$ for the different values of c are shown in Fig. II. Since α is even, a negative value of u must be considered as indicating a repulsive force. We get all types of orbits for both attraction and repulsion if we consider only $+m^2$ in the discussion of this case. As u passes through zero, that is, as the orbit passes through infinity, the force changes from attraction to repulsion. The discusssion of the sub-cases involves no new features.

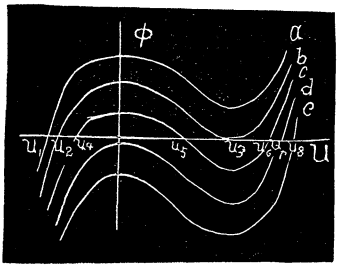


Fig. II. $\alpha>3$ and even.
Attraction and Repulsion.

The characteristics of the orbits in all cases are shown in the Table of Results, in which the italics indicate repulsive forces. The graphs are numbered to correspond to the cases. For $\alpha=3$ the graph of $\phi(u)$ and the types of orbits for repulsion are the same as in Case I, and these are not repeated. In Case V the graphs of $\phi(u)$ for both attraction and repulsion are shown in the same figure; for all negative values of u, $\phi(u)$ is imaginary.

The following are peculiarities of the orbits not already discussed.

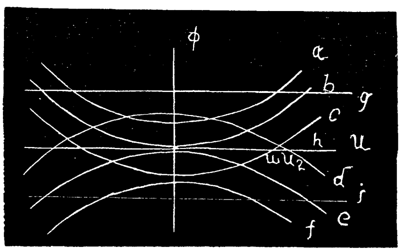


Fig. III. $\alpha=3$.
Attraction.

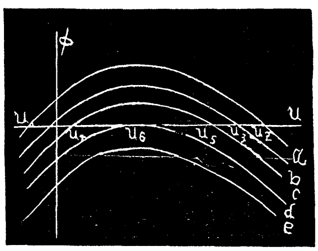


Fig. IV. $\alpha=2$.
Attraction and Repulsion.

(1) In Case III, $\alpha=3$, there are orbits which go from any finite distance to either the origin or infinity, making an infinite number of revolutions around the origin in each case.

(2) In Case IV, $\alpha=2$ (the Newtonian Law), we find for the first time an orbit having two apse distances.

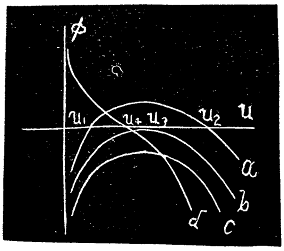


Fig. V. $\alpha=1$.
Attraction and Repulsion.

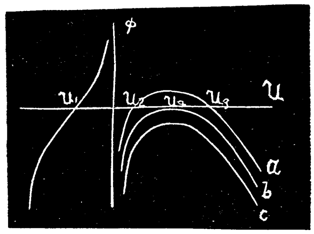


Fig. VI. $\alpha<1$ and even.
Attraction and Repulsion.

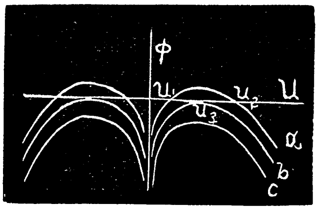


Fig. VII. $\alpha<1$ and odd.
Attraction.

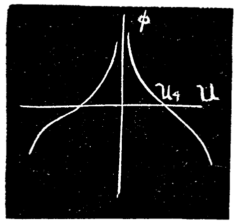


Fig. VII. $\alpha<1$ and odd.
Repulsion.

(3) In Cases V, VI, and VII, $\alpha=2$, the orbits for attraction all have two apse distances, and the orbits lie between the two circles $r=1/u_1$ and $r=1/u_2$, where u_1 and u_2 are the real roots of $\phi(u)=0$. No orbit goes to the origin or to infinity. Periodic orbits may be possible in these cases.

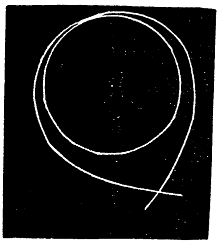


Fig. F.

(4) In Cases I, II, and III, $\alpha>2$, orbits for attraction lie outside of the region between the two circles $r=1/u_1$ and $r=1/u_2$. In Case IV, $\alpha=2$, there are orbits between two circles and also orbits which go to infinity. The Newtonian Law thus gives an intermediate case between the two general classes of orbits.

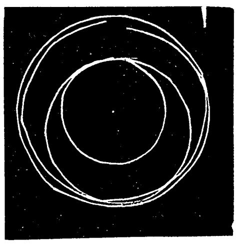


Fig. G.

(5) No orbit in any case has more than two apse distances.

(6) There are orbits which have no apse.

(7) In cases of attraction a circle is a possible orbit for all values of α.

Discussion of T. From equation (7) we get

$$t_1-t_0=\int_{u_0}^{u_1}\frac{du}{hu^2\sqrt{\phi(u)}}=\int_{u_0}^{u_1}\frac{du}{hu^2\sqrt{\pm\frac{2m^2}{a-1}u^{a-1}-u^2+c}},\quad(a\neq 1).$$

It is now shown under what conditions t_1-t_0 is finite, and these criteria are then applied to the discussion of T.

If the limits are finite the integral is finite, if the integrand is infinite of order less than unity at every point of the interval $(u_0\ u_1)$. (1) If the limits do not include $u=0$, the integral is infinite only if $\phi(u)=0$ has a multiple root between or at the limits. (2) If the limits include $u=0$, the integral is infinite unless $\sqrt{[\phi(u)]}$ is infinite of order greater than one, when it is in general finite. This is equivalent to saying that the integral is finite at $u=0$ only if $a<-1$. (3) If one limit is ∞ and the other so taken that the integrand never becomes infinite between the limits, then the integral is finite if $u^2\sqrt{[\phi(u)]}$ at $u=\infty$ is infinite of order greater than unity. In other words the integral is finite if $a>-1$.

Applying these criteria we find that in Cases I, II, III, IV, and V, for both attraction and repulsion, and in Cases VI and VII for attraction, we can make the general statement that all orbits going to infinity and all those which approach a circle asymptotically are not traversed in a finite time, but for all other orbits T is finite. In Cases VI and VII for repulsion we see from criterion (2) that T is infinite if $a\geqq -1$, but finite if $a<-1$; hence we have the interesting result that, *if a repulsive force varies directly as a power of the distance greater than unity, the particle goes to infinity in a finite time.*

TABLE OF RESULTS.

Case	Sub-Case	Number of Apses	Variation of r in One Orbit	Θ	Type of Orbit	T
I. $\alpha > 3$ and odd.	(a)	none	$0 \leqq \mid r \mid \leqq \infty$	finite	Fig. A	∞ at $r = \infty$
	(b)	two	$0 \leqq \mid r \mid \leqq 1/u_2$ $1/u_1 \leqq \mid r \mid \leqq \infty$	finite finite	Fig. B	finite ∞ at $r = \infty$
	(c)	one	$0 \leqq \mid r \mid \leqq 1/u_3$ $1/u_3 \leqq \mid r \mid \leqq \infty$	∞ at $r = 1/u_3$	Fig. C	∞ at $r = 1/u_3$ ∞ at $r = \infty$
			$r = 1/u_3$		circle	
	(d)	one	$r = \infty$			
			$0 \leqq \mid r \mid \leqq 1/u_4$	finite	Fig. D	finite
	(e)	one	$0 \leqq \mid r \mid \leqq 1/u_5$	finite	Fig. D	finite
	(f)	*one*	$1/u_6 \leqq \mid r \mid \leqq \infty$	*finite*	*Fig. E*	∞ at $r = \infty$
	(g)		$r = \infty$			
	(h)		*none*			
II. $\alpha > 3$ and even.	(a)	one	$0 \leqq \mid r \mid \leqq \infty$	finite	Fig. A	∞ at $r = \infty$
			$1/u_1 \leqq \mid r \mid \leqq \infty$	*finite*	*Fig. E*	∞ at $r = \infty$
	(b)	two	$1/u_3 \leqq \mid r \mid \leqq \infty$ $0 \leqq \mid r \mid \leqq 1/u_3$	∞ ∞	Fig. C	∞ at $r = 1/u_3$ ∞ at $r = \infty$
			$r = 1/u_3$		circle	
			$1/u_2 \leqq \mid r \mid \leqq \infty$	*finite*	*Fig. E*	∞ at $r = \infty$
	(c)	three	$1/u_5 \leqq \mid r \mid \leqq \infty$ $0 \leqq \mid r \mid \leqq 1/u_6$	finite finite	Fig. B	∞ at $r = \infty$ finite
			$1/u_4 \leqq \mid r \mid \leqq \infty$	*finite*	*Fig. C*	∞ at $r = \infty$
	(d)	two	$r = \infty$			
			$0 \leqq \mid r \mid \leqq 1/u_7$	finite	Fig. D	finite
	(e)	one	$0 \leqq \mid r \mid \leqq 1/u_8$	finite	Fig. D	finite

Case	Sub-Case	Number of Apses	Variation of r in One Orbit	Θ	Type of Orbit	T
III. $a=3$.	$m^2>1$ (a)	none	$0 \leq \vert r \vert \leq \infty$	∞ at $r=0$	Fig. A	∞ at $r=\infty$
	$m^2>1$ (b)	one	$0 \leq \vert r \vert \leq \infty$	∞ at $\begin{cases} r=0 \\ r=\infty \end{cases}$	Fig. A	∞ at $r=\infty$
	$m^2>1$ (c)	one	$0 \leq \vert r \vert \leq 1/u_1$	∞ at $r=0$	Fig. D	finite
	$m^2<1$ (d)	one	$1/u_2 \leq \vert r \vert \leq \infty$	finite	Fig. F	∞ at $r=\infty$
	$m^2<1$ (e)	one	$r=\infty$			
	$m^2<1$ (f)		none			
	$m^2=1$ (g)	none	$0 \leq \vert r \vert \leq \infty$	∞ at $r=0$	Fig. A	∞ at $r=\infty$
	$m^2=1$ (h)	∞	$\begin{cases} r=c \\ 0 \leq c \leq \infty \end{cases}$		circle	
	$m^2=1$ (i)		none			
			For repulsion, see	Case I, Repulsion.		
IV. $a=2$.	(a)	two	$1/u_2 \leq \vert r \vert \leq \infty$	finite	Fig. F	∞ at $r=\infty$
			$1/u_1 \leq \vert r \vert \leq \infty$	*finite*	*Fig. E*	∞ at $r=\infty$
	(b)	two	$1/u_3 \leq \vert r \vert \leq \infty$	finite	Fig. F	∞ at $r=\infty$
	(c)	two	$1/u_5 \leq \vert r \vert \leq 1/u_4$	finite	Fig. G	finite
	(d)	one	$r=1/u_6$		circle	
	(e)		none			
V. $a=1$.	(a)	two	$1/u_2 \leq \vert r \vert \leq 1/u_1$	finite	Fig. G	finite
	(b)	one	$r=1/u_3$		circle	
	(c)		none			
	(d)	*one*	$1/u_4 \leq \vert r \vert \leq \infty$	*finite*	*Fig. E*	∞ at $r=\infty$
VI. $a<1$ and even.	(a)	three	$1/u_3 \leq \vert r \vert \leq 1/u_2$	finite	Fig. G	finite
			$1/u_3 \leq \vert r \vert \leq \infty$	*finite*	*Fig. E*	$\begin{cases} \textit{finite}, & a<-1 \\ \infty, & a \geq -1 \end{cases}$
	(b)	two	$r=1/u_4$		circle	
			$1/u_1 \leq \vert r \vert \leq \infty$	*finite*	*Fig. E*	$\begin{cases} \textit{finite}, & a<-1 \\ \infty, & a \geq -1 \end{cases}$
	(c)	one	$1/u_1 \leq \vert r \vert \leq \infty$	*finite*	*Fig. E*	$\begin{cases} \textit{finite}, & a<-1 \\ \infty, & a \geq -1 \end{cases}$
VII. $a<1$ and odd.	(a)	two	$1/u_2 \leq \vert r \vert \leq 1/u_1$	finite	Fig. G.	finite
	(b)	one	$r=1/u_3$		circle	
	(c)		none			
	(d)	*one*	$1/u_4 \leq \vert r \vert \leq \infty$	*finite*	*Fig. E*	$\begin{cases} \textit{finite}, & a<-1 \\ \infty, & a \geq -1 \end{cases}$

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

291. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

An empty water tank has two inflow pipes A, B, which begin to flow at the same moment. When B, the smaller pipe, has discharged s gallons, and the tank is $1/n$ filled, water from both pipes is turned off. After A, B, have been idle, each as many hours as would suffice it to perform $1/m$ the work done previously by the other pipe, the flow, which is of a uniform rate, is resumed and continued till the tank is filled; B during the second working period has discharged t gallons. (1) What is the capacity of the tank? (2) What would be the capacity if B were an outflow pipe?

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let C=capacity, x=number of hours for A to fill tank, y=number of hours for B to fill tank. Then C/x=what A does in one hour, C/y=what B does in one hour. sy/C=time for B to discharge s gallons, $(C-sn)x/(Cn)$ =time for A to discharge $C/n-s$ gallons, sx/Cm=time A is idle, $(C-sn)y/(Cmn)$=time B was idle, $(s+t)y/C$=total time B works, and $(C-s-t)x/C$ =total time A works.

$$\text{But } \frac{(s+t)y}{C}=\frac{(C-s-t)x}{C}-\left(\frac{(C-sn)y}{Cmn}-\frac{sx}{Cm}\right).$$

$$\therefore \left(\frac{s+t}{C}+\frac{C-sn}{Cmn}\right)y=\left(\frac{C-s-t}{C}+\frac{s}{Cm}\right)x\ldots(1).$$

$$\text{Also } \frac{sy}{C}=\frac{(C-sn)x}{Cn}, \text{ or } y=\frac{(C-sn)x}{sn}\ldots(2).$$

(2) in (1) gives $C=mn(sn-s-t)+2sn$.

II. $\dfrac{(C+sn)x}{Cn}$=time A works before turned off.

$\dfrac{(C+s+t)x}{C}$=total time A works, $\dfrac{(C+sn)y}{Cmn}$=time B was idle.

$$\therefore\ \left(\frac{s+t}{C}+\frac{C+sn}{Cmn}\right)y=\left(\frac{C+s+t}{C}+\frac{s}{Cm}\right)x\ldots(3);\ y=\frac{(C+sn)x}{sn}\ldots(4).$$

(4) in (3) gives $C=mn(sn-s-t)-2sn$.

292. Proposed by REV. R. D. CARMICHAEL, Anniston, Ala.

Find the sum of the series $1^2+5^2+14^2+30^2+\ldots+[\frac{1}{6}n(n+1)(2n+1)]^2$.

Solution by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

The differences and the terms of this special series may be arranged as follows for the first seven terms:

$u_1=$	1^2		5^2		14^2		30^2		55^2		91^2		140^2
$u_1=$	1		25		196		900		3025		8281		19600
$\Delta^1 u_1=$	.	24		171		704		2125		5256		11319	.
$\Delta^2 u_1=$	.	.	147		533		1421		3131		6063	.	.
$\Delta^3 u_1=$	.	.	.	386		888		1710		2932	.	.	.
$\Delta^4 u_1=$	.	.	.	.	502		822		1222	.	.	.	.
$\Delta^5 u_1=$	.	.	.	.	.	320		400	.	.	.	.	.
$\Delta^6 u_1=$	.	.	.	.	.	.	80	.	.	.	.	.	.

Compute the series for ten terms, or more, and it will be found that $\Delta^6 u_1$ are all 80, or constant, therefore all the higher differences vanish. To sum the series we have the value of the leading term and the six leading differences. I have given a general formula for S_n, on page 163, of THE AMERICAN MATHEMATICAL MONTHLY for August-September, 1906, see equation (E). We have:

$$S_n=nu_1+\frac{n(n-1)}{2}\Delta^1 u_1+\frac{n(n-1)(n-2)}{3!}\Delta^2 u_1+\ldots$$

$$+\frac{n(n-1)\ldots(n-6)}{7!}n^6 u_1\ldots(1).$$

From the problem and the above table we have: $u_1=1$, $\Delta'=24$, $\Delta^2=147$, $\Delta^3=386$, $\Delta^4=502$, $\Delta^5=320$, and $\Delta^6=80$. Substitute numerical values in (1), expand the terms, consolidate like terms, reduce, and we have:

$$S_n=\frac{20n^7+140n^6+371n^5+455n^4+245n^3+35n^2-6n}{1260}\ldots(2),$$

$$=\tfrac{1}{1260}[n(n+1)(n+2)(2n+1)(2n+3)(5n^2+10n-1)].$$

Also solved by E. B. Escott, and G. B. M. Zerr. Professor Escott solved the problem by putting the general term equal to $A+Bn+Cn(n+1)+...+Gn(n+1)(n+2)(n+3)(n+4)(n+5)$. Then by letting $n=0$, -1, -2, etc., he determines A, B, ..., G. The general term is thus reduced to five terms of the form $n(n+1)...(n+r-1)$. Since the sum of a series whose general term is $n(n+1)(n+2)...(n+r-1)$ is $[n(n+1)...(n+r-1)]/[r+1]$ finds the sum which agrees with that obtained by Mr. DeLand.

Dr. Zerr decomposed the general term in a similar way and after summing the five similar series thus arising he gets the same result as that given above.

GEOMETRY.

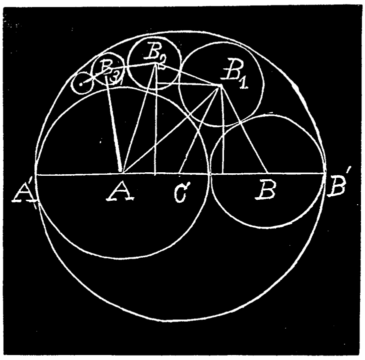
326. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

The circle C of radius pR encloses the circles A_1, B_1 of radii R and $(p-1)R$, respectively; the circle B_1 is tangent to A_1, B_1, C_1; the circle B_2 is tangent to A, B_1, C; the circle B_3 to A, B_2, C, ..., B_n to A, B_{n-1}, C. Find the radius of the circle B_n.

Solution by the PROPOSER.

First find the locus of centers of circles tangent to A and C, taking A' the point of contact of A and C as the origin.

Let r, r_1, r_2, ..., r_n be the radii of B, B_1, ..., B_n, respectively; r'=radius of any circle tangent to circles whose centers are A, C; and x, y co-ordinates of its centers. Then $(r'+R)^2-(R-x')^2=(pR-r')^2-(pR-x')^2=y'^2$...(1).



$$\therefore r=\frac{(p-1)x'}{p+1}...(2), \text{ and } x'=\frac{(p+1)r'}{p-1}...(3).$$

Substituting the value of x' in (1), we have

$$(R+r')^2-(R-\frac{p+1}{p-1}r')^2=y'^2.$$

$\therefore y'=\frac{2}{p-1}\sqrt{[p(p-1)Rr'-pr'^2]}$. Since $r=(p-1)R$ and $x=R(p+1)$, $(p-1)R$ is of the form $\frac{p(p-1)R}{0^2(p-1)^2+p}$.

2. Find r_1. Join centers of A, B, and C with B_1, B_2, ..., B_n. Draw perpendiculars from centers B_1, B_2, ..., to the diameter of C passing through A'.

$$(r+r'_1)^2-(x-x_1)^2=(pR-r_1)^2-(pR-x_1)^2...(4). \quad x_1=\left(\frac{p+1}{p-1}\right)r_1.$$

Substitute the values of x, r, x_1 in (4); whence

$$4Rr_1(p^2-p+1)=(p-1)pR^2. \quad \therefore r_1=\frac{p(p-1)R}{(p-1)^2+p}=\frac{p(p-1)R}{1^2(p-1)^2+p}.$$

(3). Find r_2. Let $(p-1)=s$. $(r_1+r_2)^2-(x_1-x_2)^2=(y_1-y_2)^2$..(5).

$$r_1^2=\left(\frac{ps}{s^2+p}\right)^2 R^2;\quad x_1^2=\frac{p^2(p+1)^2R^2}{(s^2+p)^2};\quad x_2^2=\frac{(p+1)^2}{s^2}r_2^2;\quad y_1=\frac{2pRs^2}{s(s^2+p)};$$

$$y_2=\frac{2}{s}\sqrt{[pRsr_2-pr_2^2]}.$$

Equation (5) becomes, after taking square root of both members,

$$\sqrt{\frac{(ps)^2R^2}{s^2+p}-\frac{p^2(p+1)^2R^2}{(s^2+p)^2}+\frac{2p(p+1)^2r_2R}{s(s^2+p)}+r_2^2-\frac{(p+1)^2r_2^2}{s^2}}$$

$$=\frac{2pRs^2}{s(s^2+p)}-\frac{2}{s}\sqrt{[pRsr_2-pr_2^2]}\ldots(6).$$

After clearing of radicals and reducing, (6) becomes

$$\frac{p[(s^2-p^2)+4s^2p]R^2}{(s^2+p)^2}+\frac{p+4s^2}{s^2}r_2^2-\frac{2[s^2(2p^2-p+2)+p^2]Rr_2}{s(s^2+p)}=0\ldots(7).$$

$$\therefore r_2=\frac{s[s^2(2p^2-p+2)+p^2]R}{(s^2+p)(p+4s^2)}$$

$$-\frac{R\sqrt{\{[s^2(2p^2-p+2)+p^2]^2-p(p+4s^2)(s^2+p)^2\}}}{(s^2+p)(p+4s^2)},$$

$$R\sqrt{\{[s^2(2p^2-p+2)+p^2]^2-p(p+4s^2)(s^2+p)^2\}}$$
$$=2s^2(s^2+p)=s^2(2p^2-2p+2).$$

$$\therefore r_2=\frac{Rsp(s^2+p)}{(s^2+p)(p+4s^2)}=\frac{p(p-1)R}{2^2(p-1)^2+p}.$$

3. Find r_n. Assume that $r_{n-1}=\dfrac{p(p-1)R}{(n-1)^2(p-1)^2+p}$.

Let $(n-1)=t$; then $r_{n-1}=\dfrac{pRs}{t^2s^2+p}$. $(r_t+r_n)^2-(x_t-x_n)^2=(y_t-y_n)^2\ldots(8)$.

$$x_t=\frac{p(p+1)R}{t^2s^2+p};\quad y_t=\frac{2tpsR}{t^2s^2+p};\quad y_n=\frac{2}{s}\sqrt{[psRr_n-pr_n^2]}.$$

As in case 2, (8) becomes

$$\frac{[t^2s^2+p-(p^2+1)]^2+4t^2ps^2}{(s^2t^2+p)^2}r_n{}^2-\frac{2ps\,[(t^2s^2-p)t^2s^2+p-(p^2+1)]Rr_n}{s^2t^2+p}$$

$$+\frac{4t^2(p^2+1)s^2ps}{s^2t^2+p}Rr_n+\frac{p^2s^2[\,(t^2s^2-p)^2+4t^2ps^2]R^2}{(s^2t^2+p)^2}=0\ldots(9);\ \text{whence}$$

$$r_n=\frac{psR\{(t^2s^2-p)\,[t^2s^2+p-(p^2+1)]+2t^2s^2(p^2+1)\}}{(t^2s^2+p)\,[t^2s^2+p-(p^2+1)^2+4t^2ps^2}$$

$$-[pRs\sqrt{\{(t^2s^2-p[t^2s^2+p-(p^2+1)+2t^2s^2(p^2+1)]^2} \\ -(t^2s^2+p)^2[t^2s^2+p-(p^2+1)]^2+4pt^2s^2)\}] \\ /(t^2s^2+p)\,[t^2s^2+p-(p^2+1)]^2+4t^2ps.$$

The quantity under the radical$=2t^3s^4+2tps^2$.

$$\therefore r_n=\frac{pRs\{[(t^2s^2-p)\,(t^2s^2+p-(p^2+1)]+2t^2s^2(p^2+1)-2ts^2(s^2t^2+p)\}}{(t^2s^2+p)\,[t^2s^2+p-(p^2+1)]^2+4ps^2t^2}$$

$$=[pRs\{(t^2s^2-p)\,[(t^2s^2+p-(p^2+1)]+2t^2s^2(p^2+1)]-2ts^2(s^2t^2+p)\}] \\ /\{(t^2s^2-p)\,[(t^2s^2+p-(p^2+1)]+2t^2s^2(p^2+1) \\ -2ts^2(s^2t^2+p)\}\{(t+1)^2(p-1)^2+p\}.$$

$$=\frac{psR}{(t+1)^2(p-1)^2+p}=\frac{p(p-1)R}{n^2(p-1)^2+p}.$$

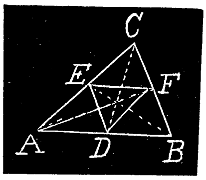
Since it has been shown that this expression is true for B_1 and B_2, it follows that it is true for B_n.

Excellent demonstrations were received from G. B. M. Zerr and C. E. White.

327. Proposed by J. C. CORBIN, Pine Bluff, Ark.

In triangle ABC, the triangle DEF is formed by joining the feet of the medians and four parallelograms are also formed, viz., $AEFD$, $BFED$, and $CEDF$. Let a, b, c; d, e, f represent the three medians of ABC, and the three sides of DEF. Then the sum of the squares of the six diagonals equals the sum of the squares of the twelve sides of the parallelograms, which are equal in sets of four. That is, $a^2+b^2+c^2+d^2+e^2+f^2=4(d^2+e^2+f^2)$, or $a^2+b^2+c^2=3(d^2+e^2+f^2)=3/4(AB^2+BC^2+CA^2)$.

Solution by J. H. MEYER, S. J., Augusta, Ga.



Let $CD=a$, $AF=b$, $EB=c$, $DF=e$, $EF=d$, and $ED=f$. Now, by geometry, we know that

a^2+d^2 in parallelogram $ECFD=2f^2+2a^2$;
b^2+f^2 in parallelogram $AEFD=2e^2+2d^2$;
c^2+e^2 in parallelogram $BFED=2f^2+2d^2$.

$$\therefore a^2+b^2+c^2+d^2+e^2+f^2=4[e^2+f^2+d^2];$$
$$\therefore a^2+b^2+c^2=3[e^2+f^2+d^2].$$

Since $e^2+f^2+d^2$ (by construction) $=\dfrac{AC^2}{4}+\dfrac{CB^2}{4}+\dfrac{AB^2}{4}$

$$\therefore a^2+b^2+c^2=\tfrac{3}{4}[AC^2+CB^2+AB^2].$$

Also solved by G. B. M. Zerr, J. Scheffer, and the Proposer.

CALCULUS.

255. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Find the general values of u and v in terms of x, which satisfy the equations $u^2+l^2(du/dx)^2=v^2$, $u^2+m^2(du/dx)^2=v^2+n^2(dv/dx)^2$.

Solution by A. F. CARPENTER, Professor of Mathematics, Hastings, Neb.; GEORGE W. HARTWELL, Columbia University, and the PROPOSER.

Subtracting the first equation from the second, we get

$$[m^2-l^2]/n^2[du/dx]^2=[dv/dx]^2.$$

$$\therefore v=\left(\frac{m^2-l^2}{n^2}\right)^{\frac{1}{2}}u+C\ldots(1).$$

$$\therefore u^2+l^2[du/dx]^2=\frac{m^2-l^2}{n^2}u^2+2u\,C\left(\frac{m^2-l^2}{n^2}\right)^{\frac{1}{2}}+C^2.$$

Let $[m^2-l^2-n^2]/n^2=a$, and $C\left(\dfrac{m^2-l^2}{n^2}\right)^{\frac{1}{2}}=b$.

Then $l^2[du/dx]^2=au^2+2bu+C^2$.

$$\therefore x=l\int\frac{du}{\sqrt{[au^2+2bu+C^2]}}=\frac{l}{2C}\log\left(\frac{au+b-C}{au+b+C}\right)+\log C_1.$$

$$\therefore x=\frac{l}{2C}\log\left(\frac{C_1[au+b-C]}{au+b+C}\right).\quad \text{Let } e^{2Cx/l}=c.$$

Then $c/C_1=\dfrac{au+b-C}{au+b+C}$, or $u=\dfrac{[b+C]c-[b-C]C_1}{a[C_1-c]}$.

$$\therefore u=\frac{\{C[(m^2-l^2)^{\frac{1}{2}}+n]/n\}e^{2Cx/l}-C_1C[(m^2-l^2)^{\frac{1}{2}}-n]/n]}{[m^2-l^2-n^2][C_1-e^{2Cx/l}]/n^2}$$

$$=\frac{Cn\{[(m^2-l^2)^{\frac{1}{2}}+n]e^{2Cx/l}-C_1[(m^2-l^2)^{\frac{1}{2}}-n]\}}{[m^2-l^2-n^2][C_1-e^{2Cx/l}]}.$$

v is found at once from (1).

Also solved by V. M. Spunar, J. Scheffer, A. H. Holmes, and J. I. Wodo.

256. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that $\sum_{x=0}^{x=\infty}\frac{x^{2m}}{1+x^{2n}}=\frac{\pi}{2n\sin\frac{2m+1}{2n}\pi}$, m and n being positive integers of which n is the greater.

I. Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

$$\sum_{x=0}^{x=\infty}\frac{x^{2m}}{1+x^{2n}}=\int_0^\infty\frac{x^{2m}dx}{1+x^{2n}}.$$

Let $x^{2n}=y$, and let $\frac{2m+1}{2n}=r$. Then

$$\int_0^\infty\frac{x^{2m}dx}{1+x^{2n}}=\frac{1}{2n}\int_0^\infty\frac{y^{r-1}dy}{1+y}=\frac{1}{2n}\Gamma(r)\Gamma(1-r)=\frac{\pi}{2n\sin r\pi}=\frac{\pi}{2n\sin\frac{2m+1}{2n}\pi}.$$

II. Solution by V. M. SPUNAR, Mechanical and Civil Engineer, East Pittsburg, Pa.

This problem belongs in a restricted sense to Algebra; as, however, the process is intimately connected with integration, let α_1, α_2, α_3, ..., α_n denote the roots of $x^{2n}+1=0$, the roots being real and unequal; hence, by the theory of equations, α is of the form

$$\alpha=\cos\frac{(2k+1)\pi}{2n}+i\sin\frac{(2k+1)\pi}{2n},$$

in which k is either 0 or a positive integer less than n $[k=0, 1, ..., (n-1)]$. Again, by the method of decomposition, we have

$$A-Bi=\frac{\alpha^{2m}}{2n\,\alpha^{2n-1}}=-\frac{\alpha^{2m+1}}{2n}.$$

$$\therefore \alpha^{2m+1}=\cos(2k+1)\phi+i\sin(2k+1)\phi,\text{ where }\phi=\frac{(2m+1)\pi}{2n}.$$

Hence, $B=\frac{\sin(2k+1)\phi}{2n}$.

$$\therefore B_1+B_2+B_3+...+B_n=\frac{1}{2n}[\sin\phi+\sin3\phi+...+\sin(2n-1)\phi]$$

$$=\frac{1}{2n}\left(\frac{1}{\sin\frac{(2m+1)\pi}{2n}}\right),$$ by a well known formula in Trigonometry.

$(B_1+B_2+B_3+\ldots+B_n)2\pi=\int_{-\infty}^{\infty}\frac{x^{2m}dx}{1+x^{2n}}$, by a well known formula in Calculus.

$$\text{Hence, } \sum_{x=-\infty}^{x=+\infty}\frac{x^{2m}}{1+x^{2n}}=\frac{\pi}{n\sin\frac{(2m+1)\pi}{2n}}=2\sum_{x=-0}^{x=+\infty}\frac{x^{2m}}{1+x^{2n}}.$$

$$\text{Hence, } \sum_{x=0}^{x=+\infty}\frac{x^{2m}}{1+x^{2n}}=\frac{\pi}{2n\sin\frac{(2m+1)\pi}{2n}}.$$

Also solved by J. I. Wodo.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

120. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Find the prime numbers p for which $x^2-pxz-px-z+p^2-3=0$ has more than two sets of positive integral solutions x, z, each $<p$.

II. Solution by E. B. ESCOTT, Ann Arbor, Mich.

The solution of this problem which was published in the March number of THE AMERICAN MATHEMETICAL MONTHLY does not seem to answer the question exactly, but rather proposes instead another problem which seems equally difficult, as there is no certainty that the numbers p so found will be prime. Thus, the solution given leads to the following values of p, 74 and 394199, neither of which is prime.

The following method will give an indefinite number of primes p which satisfy the given equation, and leads to formulas such as $p=x^2-x-1$;

$$x=n^2+n-1,\ p=n^3+n^2-2n-1. \tag{1}$$

$$\text{In the equation, } z+1=n=\frac{x^2+p^2-2}{px+1}, \tag{2}$$

p and x occur symmetrically, and n will be unchanged if p and x are interchanged. If the roots of the equation (2), considered as an equation in x, are x_1 and x_2, we have $x_2=p_1n-x_1$, and since, as just noticed, p and x may be interchanged, we may take this value of x_2 for a new p (say p_2) and take the new $x_1=p_1$. n will be unchanged. The new $x_2=p_2n-x_1=p_2n-p_1$. Let this equal p_3, etc. Then we see that in the recurring series

P_n : x_1, p_1, p_1n-x_1, ..., where the scale of relation is $P_{n+2}=nP_{n+1}-P_n$,

any two adjacent terms of the series may be taken as x and p, the initial terms x_1 and p_1 being any solution of the equation (2).

Example. Let $x_1=1$, $p_1=a$, then $n=a-1$. The series is

$$1,\ a,\ a^2-a-1,\ a^3-2a^2-a+1,\ a^4-3a^3+3a,\ a^5-4a^4+2a^3+5a^2-2a-1.$$

If we take the second and third terms for x and p, we shall have the first of the formulas given; with the third and fourth terms, we shall have the second formulas (1).

Numerical Examples.

$n=2$	1, 3, 5, 7, 9, 11, 13, ...
$n=3$	1, 4, 11, 29, 76, 199, 521, ...
$n=4$	1, 5, 19, 71, 265, 989, 3691, ...
$n=5$	1, 6, 29, 139, 666, ...
$n=6$	1, 7, 41, 239, 1393, ...

Every prime occurring in this table is a value of p and the term preceding it is the corresponding value of x. Examples of primes p with four values of x: $p=29$, $x=1, 6, 11, 27$; $p=71$, $x=1, 9, 19, 69$; $p=239$, $x=1, 16, 41, 237$.

III. **Solution by the PROPOSER.**

The following method, which is that developed by the Proposer at the time of setting the problem, will be shown to lead to all possible solutions. If $(p^2-1)^2$ has the factor $1+xp$, the complementary factor has the form $1+yp$ and there exists an integer $k\geqq 1$ (designated $z+1$ in the proposed congruence) such that

$$x+y=pk,\quad xy+k=p^2-2.$$

Eliminating y and x in turn, we get

$$k=\frac{x^2+p^2-2}{1+xp},\quad k=\frac{y^2+p^2-2}{1+yp}\ldots(1),$$

$$(p^2-1)^2=(1+xp)\,[1+(pk-x)p]=(1+yp)\,[1+(pk-y)p]\ldots(2).$$

In (1_1), x and p enter symmetrically; in (1_2), y and p. Hence

$$(x^2-1)^2=(1+px)\,[1+(xk-p)x],\quad (y^2-1)^2=(1+py)\,[1+(yk-p)y]\ldots(3).$$

If $k=1$ in (1), then from $(x-p)^2\geqq 0$ we get $p\leqq 2$. We assume that $p\geqq 2$, whence $k>1$.

Denote by the symbol $\{x,\ p\}$ a pair of numbers such that

(*i*) x and p are positive integers;

(*ii*) $x<p$;

(*iii*) $1+xp$ is a factor of $(p^2-1)^2$.

From one such pair we may derive a right-neighboring pair $\{p,\ y\}$, where $y=pk-x$. Since $x<p$, $k>1$, we have $p<y$, so that properties (*i*) and (*ii*) hold for $\{p,\ y\}$. Property (*iii*) is true by (3_2). Hence any pair leads to a chain of right-neighboring pairs:

$$\{x,\ p\},\ \{p,\ pk-x\},\ \{pk-x,\ (pk-x)k-p\}\ldots(4).$$

For $x=1$, $k=p-1$, by (1). Hence we have the successive pairs

$$\{1,\ k+1\},\ \{k+1,\ k^2+k+1\},\ \{k^2+k-1,\ k^3+k^2-2k-1\}\ldots(5).$$

When the second member of a pair (5) is a prime p, the first member is a solution x $(x<p)$. We proceed to show that every pair $\{X,\ P\}$ leading to a solution and having P prime may be obtained from the pairs (5) by assigning a suitable value to k. To this end we consider the pair $\{w,\ x\}$, $w=xk-p$, which is left-neighboring to a given pair $\{x,\ p\}$. We determine the conditions under which $\{w,\ x\}$ has the properties (*i*)—(*iii*). The third property holds in view of (3_1). By the latter,

$$wx=A-1,\ A\equiv\frac{(x^2-1)^2}{1+px},\ w=\frac{x^3-2x-p}{1+px}<\frac{x^3}{x^2},$$

since $x<p$. Hence $w<x$, and property (*ii*) holds. Finally, (*i*) holds, viz., w is positive, unless $A=0$ or 1. Hence we may form in succession left-neighboring pairs until we reach a pair with $A=0$ or 1. One of the latter cases must ultimately present itself, since a series of decreasing positive integers must terminate. If $A=0$, then $x=1$, and the chain contains the first pair (5), so that $\{X,\ P\}$ occurs in the list (5). If $A=1$, then $p=xk$, $k=x^2-2$. The first terms of the pairs (4) are then

$$x,\ x(x^2-2),\ x(x^2-2)^2-x,\ x(x^2-2)^3-2x(x^2-2),\ \ldots$$

a series of increasing integers with the factor x, so that no one is a prime P.

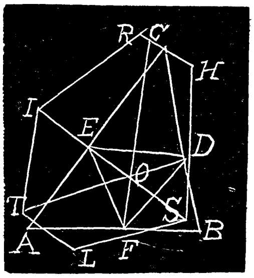
MECHANICS.

210. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

A rigid triangle is formed of three weightless, smoothly jointed, rigid rods BC, CA, AB. At their mid points D, E, F, respectively, are small, smooth rings, through which passes an endless, stretched, elastic string, forming the triangle DEF. Find by graphical construction the reaction at the joints.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let ABC be the triangle, DEF the stretched string, P the force exerted by the string, O the in-center of DEF. Then the resultant of the forces in DE, DF is $2P\cos\frac{1}{2}A$, acting from D through O. Let OT represent the this force. The resultant of the forces in FE, FD is $2P\cos\frac{1}{2}C$ acting from F through O. Let OR represent this force. The resultant of the forces in EF, ED is $2P\cos\frac{1}{2}B$ acting from E through O. Let OS represent this force. OT can be replaced by two parallel forces equal to $\frac{1}{2}OT$ acting at B and C, respectively; OR, by two parallel forces equal to $\frac{1}{2}OR$ acting at A and B, respectively; and OS, by two parallel forces equal to $\frac{1}{2}OS$ acting at A and C, respectively. Completing the parallelograms, we get OH the resultant of OR and OS, and $\frac{1}{2}OH$ represents the reaction at A. OI is the resultant of OR and OT while $\frac{1}{2}OI$ represents the reaction at B. OL is the resultant of OT and OS, and $\frac{1}{2}OL$ represents the reaction at C.



AVERAGE AND PROBABILITY.

194. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

What is the mean value of the triangle formed by joining three points taken at random on the circumference of a circle?

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let OC=diameter of given circle$=2a$. Let the point P be fixed, draw PO perpendicular to OC, and draw OA and OB. Let $\angle POA=\theta>\frac{1}{2}\pi$, $\angle POB=\phi<\frac{1}{2}\pi$.

$\therefore OA=2a\sin\theta$, and $OB=2a\sin\phi$. Area $OAB=2a^2\sin\theta\sin\phi\sin(\theta-\phi)$. Average area$=\triangle$,

$$=\frac{\int_0^\pi\int_0^\theta 2a^2\sin\theta\sin\phi\sin(\theta-\phi)\,d\theta\,d\phi}{\int_0^\pi\int_0^\theta d\theta\,d\phi}$$

$$=\frac{4a^2}{\pi^2}\int_0^\pi\int_0^\theta \sin\theta\sin\phi\sin(\theta-\phi)\,d\theta\,d\phi=\frac{2a^2}{\pi^2}\int_0^\pi(\sin^2\theta-\theta\sin\theta\cos\theta)\,d\theta$$

$$=\frac{3a^2}{2\pi}.$$

MISCELLANEOUS.

172. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

If ϕ and ψ are small angles, show that an approximate value of ϕ/ψ is

$$\tfrac{2}{3}\frac{\sin\phi}{\sin\psi}+\tfrac{1}{3}\frac{\tan\phi}{\tan\psi}-\tfrac{1}{180}(\phi^2-\psi^2)(9\phi^2-\psi^2).$$

Solution by G. W. GREENWOOD, Dunbar, Pa.

$$A=\frac{\sin\phi}{\sin\psi}=\frac{\phi}{\psi}\left(1-\frac{\phi^2}{6}+\frac{\psi^2}{6}+\frac{\phi^4}{120}-\frac{\phi^2\psi^2}{36}+\frac{7\psi^4}{360}\cdots\cdots\right)$$

$$B=\frac{\cos\phi}{\cos\psi}=1+\frac{\phi^2}{2}-\frac{\psi^2}{2}+\frac{5\phi^4}{24}-\frac{\phi^2\psi^2}{4}+\frac{\psi^4}{24}\cdots\cdots$$

$$\frac{\tan\phi}{\tan\psi}=AB=\frac{\phi}{\psi}\left(1+\frac{\phi^2}{3}-\frac{\psi^2}{3}+\frac{2\phi^4}{15}-\frac{\phi^2\psi^2}{9}-\frac{\psi^4}{45}\cdots\cdots\right)$$

$$\therefore\ \frac{2}{3}\frac{\sin\phi}{\sin\psi}+\frac{1}{3}\frac{\tan\phi}{\tan\psi}=\frac{\phi}{\psi}\left(1+\frac{\phi^4}{20}-\frac{\phi^2\psi^2}{18}+\frac{\psi^4}{180}\cdots\cdots\right)$$

$$=\frac{\phi}{\psi}[1+\tfrac{1}{180}(\phi^2-\psi^2)(9\phi^2-\psi^2)].$$

The problem does not appear correct.

Solved in a similar way and with the same result by G. B. M. Zerr.

PROBLEMS FOR SOLUTION.

ALGEBRA.

301. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

A is at Philadelphia, B at Chicago. A's personal equation is e; B's is E. When a star crosses A's meridian at time t_1=8 hours, 33 minutes, 24 seconds, he presses a button, telegraphing the fact to B, who receives it at time t_2=7 hours, 43 minutes, 23 seconds. When it crosses B's meridian at time T_2=8 hours, 33 minutes, 10 seconds, he telegraphs A, who receives it at time T_1=9 hours, 23 minutes, 11 seconds. They now exchange places, and on the second day following, B observes the transit at time t'_1=8 hours, 33 minutes, 26 seconds, and A gets the information at Chicago at time t'_2= 7 hours, 43 minutes, 25 seconds. It crosses A's meridian at time T'_2=8 hours, 33 minutes, 12 seconds, and B gets the information at time T'_1=9 hours, 23 minutes, 13 seconds. Find the difference of longitude between Philadelphia and Chicago.

302. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Prove that the system of equations

$$xu-5yv=2,$$
$$xv+yu=1,$$

has no integral solution in x, y, u, v except those for which one of the unknowns is zero.

GEOMETRY.

335. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Determine analytically, the point where three lines in a plane appear of equal length.

CALCULUS.

260. Proposed by V. M. SPUNAR, East Pittsburg, Pa.

A natural equation of a surface may be defined as an equation in which the differential quotients of the principal radius, ρ, of curvature to the element of arc in the direction of the principal curvature are shown as a function of ρ, $\dfrac{d^n\rho}{ds}=F(\rho)$. Required the natural equation of the whole surface of second power.

261. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that $\displaystyle\sum_{x=1}^{x=\infty}\frac{1}{a+2bx^2+cx^4}=\frac{\pi}{\sqrt{[8ac(\sqrt{ac}+b)]}}-\frac{1}{2a}$, where $ac>b^2$.

262. Proposed by H. SCHAFFER, Fayetteville, Ark.

Prove that the circle is the only plane curve of constant curvature.

MECHANICS.

217. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Given, the mean distance from earth to sun, 1.49×10^{15} centimeters; radius ef the earth, 6.37×10^8 centimeters; velocity of the earth in its orbit, 2.96×10^6 centimeters per second; velocity of rotation of a point on the equator, 4.63×10^4 centimeters per second; mass of the earth, 6.14×10^{27} grams; find (1) the total energy of the earth in ergs; (2) the angular velocity of the earth on its axis; and (3) the angular velocity of the earth around the sun.

218. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Cut a uniform, circular cylinder by two planes whose line of intersection is without the cylinder. The centroid G of the surface of the portion of the cylinder thus cut off lies in a plane elliptic section, in which plane also lies the line of intersection aforesaid. C is the center of the ellipse, and the pole of the intersection line with reference to this ellipse is X. Show (1) that C, X, and G are collinear, and (2) that $XC=2CG$.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

153. Proposed by LLOYD HOLSINGER, A. B., 227 Fredonia Avenue, Peoria, Ill.

If we represent by (k, l) the greatest common divisor of k and l, and by $\phi(k)$ the number of integers prime to k and not greater than k, we have

$$\begin{vmatrix} (1,1) & (1,2) & (1,3) & \cdots & (1,n) \\ (2,1) & (2,2) & (2,3) & \cdots & (2,n) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (n,1) & (n,2) & (n,3) & \cdots & (n,n) \end{vmatrix} = \phi(1).\phi(2).\phi(3)\ldots.\phi(n).$$

AVERAGE AND PROBABILITY.

198. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Find the average length of a hole at random through a given cylinder.

MISCELLANEOUS.

178. Proposed by V. M. SPUNAR, Mechanical and Civil Engineer, East Pittsburg, Pa.

Find the sum of the series, $\dfrac{\sin x}{m^2+1}-\dfrac{2\sin 2x}{m^2+4}+\dfrac{3\sin 3x}{m^2+9}-\ldots.$ to infinity.

ERRATA.

Page 100, line 2 from bottom, for "(t_m, s_m)" read $\rho(t_m, s_m)$.

Page 101, line 1 from top, for "positive" read negative; for "negative" read positive.

Page 101, line 12 from top, for "$(t, \rho(t, P(t, r)))$" read $\rho(t, P(t, r))$.

Page 101, line 13 from bottom, for "I" read δ in each case.

Page 102, line 4 from top, for "modulo 1" read modulo l.

Page 102, line 14 from top, for "$\rho(t, s)>r_2$" read $\rho(t, s)\geqq r_2$.

Page 102, lines 8, 9, and 11 from bottom, the number (3) should go with the equation of line 11, and (4) should go with the equations of lines 8 and 9, which should be braced.

Page 102, line 8 from bottom, for "$y(t+h)-\lambda x'(t+h)$" read $y(t+h)+\lambda x'(t+h)$.

Page 103, line 10 from bottom, for "$x(t+\bar{h})-y'(t+\bar{h})$" read $x(t+\bar{h})-\bar{\lambda} y'(t+\bar{h})$.

Page 103, line 3 from bottom, for "π" read ρ.

NOTE.

The radicals on pp. 76-77 are connected by the relation

$$8\sqrt{C_\pm}=\sqrt{D_+}.(a-\sqrt{B})\pm\sqrt{D_-}.(a+\sqrt{B}).$$

This relation invalidates a criticism of the accuracy of the results found.

L. E. D.

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ON MATHEMATICAL INDUCTION.

By DR. J. W. A. YOUNG, The University of Chicago.

I. Its Function and Place.

In the secondary school, the pupil in mathematics is becoming familiar with this fundamental type of thought by working in it and applying it. He thinks mathematics, but not *about* mathematics. In college, when the student begins to philosophize, he may, in addition to working in and with mathematics, also begin to think about it, to analyze and classify its processes of thought, to seek its essential characteristics and the lines of demarcation between it and other subjects.

He will learn that many mathematicians see the distinctive marks of their science, not in its subject matter, not in numbers, points, lines and symbols, but in the mode of thinking which is used.* He will find the definition, "Mathematics is the science which draws necessary conclusions,"† a good expression of this conception of the subject. It identifies mathematics with deductive reasoning, and accounts for the peculiar certainty and accuracy which, in his experience, has been its distinctive characteristic.

But when he thinks of the way in which he works out mathematics, solves problems, does "original" work of any description, he sees that induction plays an important part in this aspect of mathematics.

Further, as his acquaintance with the subject matter of mathematics widens, he will find numerous instances in which concepts are extended and theorems generalized. He will frequently find that the course of development is from the particular to the general, and in such cases he will often find employed a method of reasoning called *mathematical induction*,‡ which shares with non-mathematical induction the peculiarity of generalizing from particular instances, but which nevertheless, like other mathematics, produces that unhesitating confidence in the absolute accuracy of the result

*For an instructive discussion of various definitions of mathematics, see Bocher, *Bulletin of American Mathematical Society*, 1904, p. 115.

†Peirce, *American Journal of Mathematics*, Vol. IV.

‡Also, *Complete Induction*, and, in German, *der Kaestnerische Schluss*, although published by Pascal in 1662 (Cantor, *Gesch. d. Math.*, II. p. 684) long before the time of Kaestner (1719—1800).

which is not felt as to the results of non-mathematical reasoning. This two-fold property leads one of the most acute thinkers of the day* to see in mathematical induction the sole instrument whereby the mathematician enlarges the sum-total of mathematical knowledge (at least in pure mathematics or arithmetic, as distinguished from geometry and infinitesimal analysis). Whether or not we concur† without reserve in Poincaré's thesis that "we can advance only by means of mathematical induction, which alone can teach us anything new," all will agree that explicitly it produces a large body of the mathematician's most valued results, and that implicitly it lurks unsuspected at the bottom of the reasoning by which a far greater body of mathematical truth is established.

The process of mathematical induction is exceptionally well fitted to introduce the beginner to the philosophic study of mathematical thinking, since it deals neither with the delicate and abstract questions relative to the foundations of mathematics, whose successful treatment requires extensive experience in mathematical reasoning, nor with such concepts from the borderland of mathematics and philosophy as have in the past proved themselves vague and elusive (for example, "continuity" or "infinity"). While a careful discussion of the logical nature and function of mathematical induction should perhaps be deferred to a period later than the early collegiate years, practical acquaintance with the process itself and a certain amount of readiness in its use may well be acquired at that time. To this end, a considerable body of material is requisite, both to illustrate the range and fertility of the method and also to supply a fund of exercises sufficiently ample to meet the student's need of considerable practice, and to permit variety in the same class and in successive classes.

II. Its Character.

I give some of the exercises which I have collected for such use from many sources, preceded by a discussion of the method itself based on a particular example.

By trial:

$$\begin{aligned} 1&=1,\\ 1+2&=3,\\ 1+2+3&=6,\\ 1+2+3+4&=10,\\ 1+2+3+4+5&=15. \end{aligned}$$

Letting S_n denote the sum of the first n positive integers, the equations can be written:

$$S_1=\frac{1.2}{2}‡,\quad S_2=\frac{2.3}{2},\quad S_3=\frac{3.4}{2},\quad S_4=\frac{4.5}{2},\quad S_5=\frac{5.6}{2}.$$

*Poincare, *Sur la nature du raisonnement mathematique, Revue de metaphysique et morale,* 1894, p. 371.

†I have discussed elsewhere (*The Teaching of Mathematics*, p. 25, note) the manner in which deductive reasoning enlarges the sum-total of mathematical knowledge.

‡ 1.2 means 1×2.

Examining these equations, we see that they are all of the type:

$$S_n=\frac{n(n+1)}{2}.$$

By taking a few additional ones, we find that the formula continues to hold. Thus:

$$S_6=1+2+3+4+5+6=21=\frac{6.7}{2}.$$

Trial of additional instances increases the "moral" certainty that the formula is true for every positive integer n, but no matter how many instances we may have the patience to try, mathematical certainty is not achieved thereby. This is attained by means of the mind's power of operating with mathematical certainty upon unspecified numbers.

The invariable method of mathematical induction is to prove that whenever the formula in question holds in any particular instance, it also holds in the next following instance.*

Recurring to our example, we must show that if the formula $S_n=\frac{n(n+1)}{2}$ holds for any particular value of n, say $n=k$, it also holds for the next value of n or $k+1$.

(A) That is, we must show that, whenever $S_k=\frac{k(k+1)}{2}$ holds, then, $S_{k+1}=\frac{(k+1)(k+2)}{2}$ holds also.

Proof. By definition, $S_{k+1}=S_k+k+1$.

Substituting the assumed value of S_k,

$$S_{k+1}=\frac{k(k+1)}{2}+k+1=\frac{(k+1)(k+2)}{2}.$$

The proof required above is thus made, but the result is purely hypothetical. We pass to actually valid results as follows.

(B) Trial has already shown that the formula is true for S_1 to S_6. Since the formula is true for S_6, we know, without trial but solely by the proof at (A), that the formula holds for S_7. since it is true for S_7, we know similarly without trial, that it is true for S_8. Similarly we know that the formula is true for S_9, for S_{10}, and so on to any S_n whatever.

The success of the proof hinges upon three things:

1. The ability to express any instance in terms of the next preceding, even when the latter is not specified.

*The application of the method presupposes that the cases to be considered have in some way been arranged in a definite order and numbered consecutively, and that after each case follows another.

2. The ability to make the proof (A) without specifying the particular instance that is being considered.

3. The power of the mind to see with certainty that repetition of the steps at (B) would lead to any given n, without actually going through all these steps.

These three things are but various phases of the mind's power of operating with certainty upon unspecified numbers.

In any proof by mathematical induction, the following two parts must unfailingly be present:

1). The proof that *if* the statement holds in any particular instance, it also holds in the next, and

2). The proof that the statement actually holds in the first instance (that is, in *some* particular case).

These two proofs are quite independent and may therefore be made in either order, but the relation to true induction is most apparent when the formula to be proved is actually discovered by true induction (as in the example above), thus amply making the second proof at the outset.

Beginners are prone to regard one or the other of these proofs as sufficient in itself. The need of both may be made clear by non-mathematical illustrations and by mathematical examples.

As a non-mathematical illustration we may consider a row of bricks so arranged that whenever any brick is knocked over, it will in its fall knock over its neighbor on the right. But this is only potential. In order actually to knock over the whole row, it is necessary and sufficient actually to knock over the first brick. If this cannot be done the whole row cannot be knocked over.

A second illustration is that of a ladder.* "We must have a ladder by which to climb from any round (the kth) to the next round (the $k+1$st); but the ladder must rest on a solid basis so that we can get on to the ladder (the $k=1$ or $k=2$ rounds)."

Mathematical examples can also be given in which one of the parts can be proved but not the other; and hence the statement in question is not proved.

1. Consider the series,

$$1.1!,\ 2.2!,\ 3.3!,\ \ldots.$$

It is readily proved that if the sum of the first k terms is $(k+1)!$, then the sum of the first $k+1$ terms is $(k+1+1)!$ Or, denoting the sum of the first n terms of any series by S_n, if, for this series, the formula $S_n=(n+1)!$ holds for any particular value of n, it also holds for the next following value of n. But there is no value of n for which it can be proved to hold, and therefore the formula is not proved. (If it could be proved for any partic-

*Dickson, *College Algebra*, p. 100.

ular value of n after the first, the formula would of course be proved from that value of n on.)

2. Considering the series,

$$1,\ \tfrac{3}{2},\ \tfrac{5}{4},\ \tfrac{7}{8},\ \tfrac{9}{16},\ \ldots.$$

it can be proved that if the formula

$$S_n=1-\frac{2n+3}{2^{n-1}}$$

holds for any particular value of k, it also holds for $k+1$. But no value k can be found for which the formula holds and hence it is not proved.

On the other hand it may be possible to verify a certain statement in many consecutive instances, without leading to the conclusion that it is true in every instance.

1. The expression, $1-2^{n+2}+2.3^{n+1}-4^{n+1}+5^n$ is zero for $n=0$, 1, 2, 3, but not zero for $n=4$.

2. $(4^n-6.3^n+14.2^n-14)[1-(-1)^n]$ is zero for $n=0, 1, \ldots, 6$, but not zero for $n=7$.

3. n^2+n+17 is prime for $n=0, 1, 2, \ldots, 16$, but not for $n-17$.

4. $2n^2+29$ is prime for $n=0, 1, 2, \ldots, 28$, but not for $n=29$.

5. n^2-n+41 is prime for $n=0, 1, 2, \ldots, 40$, but not for $n=41$.

6. The theorem that if an odd prime be increased by 3, the result is the product of an odd prime and a power of two, holds for the odd primes to 43, but does not hold for 47.

III. Exercises.

In the exercises that follow, a formula for S_n is to be proved, unless otherwise specified. The finding of the formula by true induction is one of the most fascinating activities, and offers, in proportion to the difficulty of the task, more or less opportunity for the application of mathematical ability and the enjoyment of mathematical inspiration. Not to rob the reader of the possibility of the pleasures, the answers are not as a rule given with the exercises, but are collected at the end.

The exercises are not arranged in order of difficulty. This will naturally vary with different minds. Of those whose answers are reserved, the following are among the easiest, arranged somewhat in order of supposed difficulty: 21, 10, 13, 1, 32, 23, 12, 11, 4, 9. The degree of difficulty of the exercises may be diminished by giving a part of the result. Thus, No. 3 becomes very easy if it is given that $4n^2+6n-1$ is a factor of the result, and No. 7 becomes moderately easy if it is given that $2n+1$ is a factor of the result.

In each of the following the sum of the series is to be found by mathematical induction.

(1) $1.2,\ 2.3,\ 3.4,\ \ldots$
(2) $1.2,\ 3.4,\ 5.6,\ \ldots$
(3) $1.3,\ 3.5,\ 5.7,\ \ldots$
(4) $2.3,\ 4.6,\ 6.8,\ \ldots$
(5) $1^2,\ 2^2,\ 3^2,\ \ldots$
(6) $1^2,\ 3^2,\ 5^2,\ \ldots$
(7) $2^2,\ 4^2,\ 6^2,\ \ldots$
(8) $4^2,\ 7^2,\ 10^2,\ 13^2,\ \ldots$
(9) $0,\ 2.3,\ 3.8,\ 4.15,\ \ldots,\ k(k^2-1),\ \ldots$
(10) $1^3,\ 2^3,\ 3^3,\ \ldots$
(11) $1^3,\ 3^3,\ 5^3,\ \ldots$
(12) $2^3,\ 4^3,\ 6^3,\ \ldots$
(13) $2.3.4,\ 3.4.5,\ 4.5.6,\ \ldots$
(14) $1.3.5,\ 3.5.7,\ 5.7.9,\ \ldots$
(15) $1.1^2,\ 2.3^2,\ 3.5^2,\ 4.7^2,\ \ldots$
(16) $1.2^2,\ 2.3^2,\ 3.4^2,\ 4.5^2,\ \ldots$
(17) $1,\ 4,\ 10,\ 20,\ \ldots,\ \dfrac{k(k+1)(k+2)}{3!},\ \ldots$
(18) $1^4,\ 2^4,\ 3^4,\ 4^4,\ 5^4,\ \ldots$
(19) $1.2.3.4,\ 2.3.4.5,\ 3.4.5.6,\ \ldots$
(20) $1,\ 5,\ 15,\ 35,\ \ldots,\ \dfrac{k(k+1)(k+2)(k+3)}{4!},\ \ldots$
(21) $\dfrac{1}{1.2},\ \dfrac{1}{2.3},\ \dfrac{1}{3.4},\ \ldots$
(22) $\dfrac{1}{1.3},\ \dfrac{1}{2.4},\ \dfrac{1}{3.5},\ \ldots$
(23) $\dfrac{1}{3.5},\ \dfrac{1}{5.7},\ \dfrac{1}{7.9},\ \ldots$
(24) $\dfrac{1}{1.2.3},\ \dfrac{1}{2.3.4},\ \dfrac{1}{3.4.5},\ \ldots$
(25) $\dfrac{1}{1.2.3},\ \dfrac{3}{2.3.4},\ \dfrac{5}{3.4.5},\ \ldots$
(26) $\dfrac{1}{2.3.4},\ \dfrac{2}{3.4.5},\ \dfrac{3}{4.5.6},\ \ldots$
(27) $\dfrac{1}{1.3.5},\ \dfrac{1}{3.5.7},\ \dfrac{1}{5.7.9},\ \ldots$
(28) $\dfrac{1}{2.5.8},\ \dfrac{2}{5.8.11},\ \dfrac{3}{8.11.14},\ \ldots$
(29) $\dfrac{1}{1.2.3.4},\ \dfrac{1}{2.3.4.5},\ \dfrac{1}{3.4.5.6},\ \ldots$
(30) $1^5,\ 2^5,\ 3^5,\ 4^5,\ \ldots$
(31) $1^6,\ 2^6,\ 3^6,\ 4^6,\ \ldots$*
(32) $1,\ 2.2,\ 3.2^2,\ 4.2^3,\ \ldots$
(33) $1,\ 2.3,\ 3.3^3,\ 4.3^3,\ \ldots$

*On the series $1^n,\ 2^n,\ 3^n,\ \ldots$ see Chrystal's *Alpha*, Vol. I, p. 486.

(34) $1,\ 2.5,\ 3.5^2,\ 4.5^3,\ \ldots$
(35) $1,\ 3.2,\ 5.2^2,\ 7.2^2,\ \ldots$
(36) $1,\ 4.3,\ 7.3^2,\ 10.3^3,\ \ldots$
(37) $1,\ 5.4,\ 9.4^2,\ 13.4^3,\ \ldots$
(38) $1,\ 4.2,\ 9.2^2,\ 16.2^3,\ \ldots$
(39) $1,\ 4.3,\ 9.3^2,\ 16.3^3,\ \ldots$
(40) $1,\ 8.2,\ 27.2^2,\ 64.2^3,\ \ldots$
(41) $1,\ 3.2,\ 6.2^2,\ 10.2^3,\ \ldots,\ \frac{k(k+1)}{2}2^{k-1},\ \ldots$
(42) $1,\ \frac{4}{3},\ \frac{9}{3^2},\ \frac{16}{3^3},\ \ldots$
(43) $a,\ 2(a+1),\ 3(a+2),\ 4(a+3),\ \ldots$

Show that:

(44) $2.6.10.14, \ldots, (4n-6)(4n-2)=(n+1)(n-2), \ldots, (2n-1)2n.$
(45) $n(n+1)(n+2), \ldots, (2n-2)=1.3.5.7, \ldots, (2n-3).2^{n-1}.$
(46) $(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots(1+x^{2^n})=1+x+x^2+\ldots+x^{2^{n+1}-1}$
(47) Show that $x^{2n}-y^{2n}$ is divisible by $x+y$.
(48) Show that $n^{13}-n$ is divisible by 13, for every positive integer n.
(49) If p is a prime number, show that n^p-n is divisible by p for every positive integer n.
(50) Show that $2.7^n+3.5^n-5$ is divisible by 24 for every positive integer n.

Suggestion: Assume, 1) $2.7^k+3.5^k-5=$multiple of 24.
To show, $2.7^{k+1}+3.5^{k+1}=$multiple of 24.
Multiply 1) by 7, $2.7^{k+1}+3.5^k.7-5.7=$multiple of 24.
Hence, $2.7^{k+1}+3.5^{k+1}-5-30+3.5.2^k=$multiple of 24.
To prove the last assertion, we must show that $6.5^k-30=$multiple of 24, which is easily done.

(51) Show that $3.5^{2n+1}+2^{3n+1}$ is divisible by 17 for every positive integer n.
(52) If n is a positive integer, show that $(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{2!}a^{n-2}b^2+\ldots+\frac{n(n-1)(n-2)\ldots(n-k+1)}{h!}a^{n-k}b^k+\ldots$
(53) If all the positive integers of n and fewer digits be written, show that the number of times any digit (other than zero) occurs is $n.10^{n-1}$.
(54) If the positive integers are grouped as follows: [(1, 2), (3)]; [(4, 5, 6), (7, 8)]; [(9, 10, 11, 12), (13, 14, 15)]; ... and these groups taken in pairs as indicated, prove that the sum of the numbers in the two groups of any pair is the same.
(55) $\sum_{k=1}^{n}(a+kb)^2=n[a^2+ab(n+1)+\frac{b^2}{6}(2n^2+3n+1)].$
(56) $\sum_{k=1}^{n}\frac{(2k)!}{k!\,2^k}=1.3.5\ldots(2n-1).$

(57) Letting t_k denote the kth term of a series, show that, if $t_k=k(k+1)\ldots(k+q-2)$, $S_n=\dfrac{n\,(n+1)\ldots(n+q-1)}{q}$.

(58) If $t_k=\dfrac{1}{k(k+1)\ldots(k+p)}$, $S_n=\dfrac{1}{p}\left[\dfrac{1}{p!}-\dfrac{1}{(n+1)\ldots(n+p)}\right]$.

(59) Assuming that the formulas,

$$\sin(x\pm y)=\sin x\,\cos y\pm\cos x\,\sin y,$$
$$\cos(x\pm y)=\cos x\,\cos y\pm\sin x\,\sin y,$$

have been proved for all acute angles x, y, prove that they hold for all posi-angles x, y. (In this proof, the quadrants are to be regarded as numbered consecutively, 5, 6, 7, 8, etc., as the angle increases beyond 360°, and the formulas of the type $\sin(90+x)=\cos x$, are to be accepted as proved for every angle x.)

(60) $(\cos a+i\sin a)^n=\cos na+i\sin na$. DeMoivre's formula.

Sum the following:

(61) $\sin a$, $\sin 2a$, $\sin 3a$, ...

(62) $\sin a$, $\sin 3a$, $\sin 5a$, ...

(63) $\cos a$, $\cos 2a$, $\cos 3a$, ...

(64) $\cos a$, $\cos 3a$, $\cos 5a$, ...

IV. Answers.

(1) $\dfrac{n(n+1)(n+2)}{3}$.

(2) $\dfrac{n(n+1)(4n+1)}{3}$.

(3) $\dfrac{n(4n^2+6n-1)}{3}$.

(4) $\dfrac{4n(n+1)(n+2)}{3}$.

(5) $\dfrac{n(n+1)(2n+1)}{6}$.

(6) $\dfrac{n(2n-1)(2n+1)}{6}$.

(7) $\dfrac{2n(n+1)(2n+1)}{3}$.

(8) $\dfrac{n(6n^2+15n+11)}{2}$.

(9) $\dfrac{(n-1)n(n+1)(n+2)}{4}$.

(10) $\left[\dfrac{n(n+1)}{2}\right]^2$.

(11) $n^2(2n^2-1)$.

(12) $2n^2(n+1)^2$.

(13) $\dfrac{(n+1)(n+2)(n+3)}{3}$.

(14) $n(2n^3+8n^2+7n-2)$.

(15) $\dfrac{n(n+1)(6n^2-2n-1)}{6}$.

(16) $\dfrac{n(n+1)(n+2)(3n+5)}{12}$.

(17) $\dfrac{(n+3)!}{4!(n-1)!}$.

(18) $\dfrac{n(n+1)(6n^3+9n^2+n-1)}{30}$.

(19) $\dfrac{n\,(n+1)(n+2)(n+3)(n+4)}{5}$.

(20) $\dfrac{(n+4)!}{5!(n-1)!}$.

(21) $\dfrac{n}{n+1}$.

(22) $\dfrac{n(3n+5)}{4(n+1)(n+2)}$.

(23) $\dfrac{n}{3\,(2n+3)}$.

(24) $\frac{1}{2}\left[\frac{1}{2}-\dfrac{1}{(n+1)(n+2)}\right]$.

(25) $\frac{n(3n+1)}{4(n+1)(n+2)}$. (26) $\frac{n(n+1)}{4(n+2)(n+3)}$.

(27) $\frac{n(n+2)}{3(2n+1)(2n+3)}$. (28) $\frac{n(n+1)}{4(3n+2)(3n+5)}$.

(29) $\frac{1}{3}\left[\frac{1}{3!}-\frac{1}{(n+1)(n+2)(n+3)}\right]$. (30) $\frac{n^2(n+1)^2(2n^2+2n-1)}{3.4}$.

(31) $\frac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{6.7}$.

(32) $(n-1)2^n+1$. (33) $\frac{(2n-1)3^n+1}{4}$.

(34) $\frac{(4n-1)5^n+1}{16}$. (35) $(2n-3)2^n+3$.

(36) $\frac{(6n-7)3^n+7}{4}$. (37) $\frac{(12n-13)4^n+13}{9}$.

(38) $(n^2-2n+3)2^n-3$. (39) $\frac{(n^2-n+1)3^n-1}{2}$.

(40) $[(n-1)^3-6(n-2)]2^n+13$. (41) $2^{n-1}(n^2-n+2)-1$.

(42) $\frac{3^{n+1}-(n^2+3n+3)}{2.3^{n-1}}$. (43) $\frac{n(n+1)(3a+2n-2)}{6}$.

(61) $\frac{\sin\frac{na}{2}\sin\frac{(n+1)a}{2}}{\sin\frac{a}{2}}$. (62) $\frac{\sin^2 na}{\sin a}$.

(63) $\frac{\sin\frac{na}{2}\cos\frac{(n+1)a}{2}}{\sin\frac{a}{2}}$. (64) $\frac{\sin 2na}{2\sin a}$.

ON THE GENERAL TANGENT TO PLANE CURVES.*

By PROF. R. D. CARMICHAEL, Anniston, Alabama.

The object of this note is to work out without the Calculus a certain well known formula for a tangent to a plane algebraic curve at an ordinary (not a singular) point, and especially to show how this result is easily extended to the loci of transcendental equations. The formula found is readily developed by aid of the Differential Calculus, but is here found by other means. It might therefore be used in a course which does not presuppose the Calculus.

Let us take, in rectangular Cartesian coordinates, the general equation of a proper nth degree locus in the form

*Read before the Chicago Section of the American Mathematical Society, April 18, 1908.

[1] $$a_{00}+a_{10}x+a_{01}y+a_{20}x^2+a_{11}xy+a_{02}y^2+\ldots+\sum_{s=0}^{s=n}a_{n-s,\,s}\,x^{n-s}y^s=0.$$

Let $[\alpha, \beta]$ be an ordinary point of the curve, so that at this point only one tangent to the curve can be drawn. Transform the equation by the substitution

[2] $$x=x'+\alpha, \quad y=y'+\beta.$$

The new origin will be on the curve; and therefore the independent term vanishes. The resulting equation takes the form

[3] $$a_{10}x'+a_{01}y'+2a_{20}\alpha\, x'+a_{11}\beta\, x'+a_{11}\alpha\, y'+2a_{02}\beta\, y'+\ldots$$
$$+\sum_{s=0}^{s=n}a_{n-s,\,s}\{[n-s]\alpha^{n-s-1}\beta^s x'+s\,\alpha^{n-s}\beta^{s-1}y'\}$$
$$+\text{terms of second degree and higher in } x',\ y'=0.$$

Let $u^{(p)}$ be identically equal to the sum of all the terms of degree p in x', y' in this equation. Then the equation takes the form

[3_a] $$u^{(1)}+u^{(2)}+u^{(3)}+\ldots+u^{(n)}=0.$$

Then is $u^{(1)}=0$ the equation of the tangent to this curve at the origin. (See Salmon-Fiedler, *Hoeheren ebenen Kurven*, 2nd ed., p. 31.) For since the origin is now an ordinary point of the curve, $u^{(1)}$ is not identically equal to zero. [If the origin were a double point $u^{(1)}$ would be identically equal to zero. Then in order to find the equations of the tangents it would be necessary to deal with $u^{(\sigma)}$, where $u^{(\sigma)}$ is the first of the quantities $u^{(2)}$, $u^{(3)}$, ... which is not identically zero. See Salmon-Fiedler, *l. c.* This case is excluded from the present discussion by the first assumption that (α, β) is an ordinary point of the locus of [1]].

Now from [3] it is easy to write

$$u^{(1)}\equiv\sum_{n=1}^{n=n}\sum_{s=0}^{s=n}a_{n-s,\,s}\{[n-s]\alpha^{n-s-1}\beta^s x'+s\,\alpha^{n-s}\beta^{s-1}y'\}=0;$$

or,

$$\frac{y'}{x'}=-\frac{\sum_{n=1}^{n=n}\sum_{s=0}^{s=n}a_{n-s,\,s}[n-s]\alpha^{n-s-1}\beta^s}{\sum_{n=1}^{n=n}\sum_{s=0}^{s=n}a_{n-s,\,s}\,s\,\alpha^{n-s}\beta^{s-1}},$$

as the equation of the tangent to the locus of [3] at the origin. Hence by [2], the tangent to the locus of [1] at the point $[\alpha, \beta]$ is

$$[4] \qquad \frac{y-\beta}{x-\alpha} = -\frac{\sum\limits_{n=1}^{n=n}\sum\limits_{s=0}^{s=n} a_{n-s,\,s}\,[n-s]\,\alpha^{n-s-1}\beta^{s}}{\sum\limits_{n=1}^{n=n}\sum\limits_{s=0}^{s=n} a_{n-s,\,s}\,s\,\alpha^{n-s}\beta^{s-1}}.$$

EXAMPLE 1. Applying this to the circle $x^2+y^2=r^2$, we have for its tangent at $[\alpha, \beta]$

$$\frac{y-\beta}{x-\alpha} = -\frac{2\alpha}{2\beta} = -\frac{\alpha}{\beta}.$$

Since $\alpha^2+\beta^2=r^2$ this reduces to the usual formula

$$x\alpha+y\beta=r^2.$$

EXAMPLE 2. In the conic $a_{20}x^2+a_{11}xy+a_{02}y^2+a_{10}x+a_{01}y+a_{00}=0$, we have

$$\frac{y-\beta}{x-\alpha} = -\frac{2a_{20}\alpha+a_{11}\beta+a_{10}}{2a_{02}\beta+a_{11}\alpha+a_{01}}$$

for the tangent at $[\alpha, \beta]$. Since $[\alpha, \beta]$ is on the locus, it is easy to reduce this equation to the form

$$[2a_{20}\alpha+a_{11}\beta+a_{10}]x+[2a_{02}\beta+a_{11}\alpha+a_{01}]y+a_{10}\alpha+a_{01}\beta+2a_{00}=0,$$

the general equation of a tangent to a conic.

Now, in the equation

$$y-\sin x\cos y=0$$

suppose that $\sin x$ and $\cos y$ are replaced by their expansions in series and that the product is taken. The resulting equation with infinite number of terms, may evidently be dealt with by the same method as that which we have used with reference to [1]. And so of the locus of any such equation. Instead of formula [4] it is evident that we should in these cases have the tangent at $[\alpha, \beta]$ determined by the following equation:

$$[5] \qquad \frac{y-\beta}{x-\alpha} = -\frac{\sum\limits_{n=1}^{n=\infty}\sum\limits_{s=0}^{s=n} a_{n-s,\,s}\,[n-s]\,\alpha^{n-s-1}\beta^{s}}{\sum\limits_{n=1}^{n=\infty}\sum\limits_{s=0}^{s=n} a_{n-s,\,s}\,s\,\alpha^{n-s}\beta^{s-1}},$$

provided both infinite series are convergent.

If the equation of the given locus is $y-f(x)=0$ where f denotes an algebraic function of x or an infinite convergent series of terms containing x only in positive integral powers and if the coefficients are represented by the same quantities as before; then equations [4] and [5] become

$$[6] \qquad \frac{y-\beta}{x-\alpha}=-\sum_{n=1}^{n=n} a_{n,\,0}\, n\, \alpha^{n-1}$$

where the range of n is finite or infinite according as $f(x)$ is or is not an algebraic function of x. In the latter case, of course, the series of [6] must be convergent for the given value of α in order that the formula may be employed.

It may be pointed out that formulae [4], [5], [6] can be obtained by differentiation and the substitution of α, β in the results. Thus if $u=0$ is the equation of the curve, these become, respectively,

$$\frac{y-\beta}{x-\alpha}=-\frac{D_xu}{D_yu};\quad \frac{y-\beta}{x-\alpha}=-\frac{D_xu}{D_yu};\quad \frac{y-\beta}{x-\alpha}=-D_xu,$$

where x and y are replaced by α and β in every D_xu and D_yu. Hence, the interest which attaches to the discussion in this note is not in the results themselves but in the method of obtaining them without the aid of the Calculus.

EXAMPLE 3. What is the tangent to the curve $y=\sin x$ at the point $[\alpha, \beta]$?

We suppose that this problem is assigned to a pupil who is yet unacquainted with the Calculus, but one who knows from Trigonometry the series

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots,$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots.$$

The given equation becomes

$$y=x-\frac{x^2}{3!}+\frac{x^5}{5!}-\ldots,$$

and therefore by [6] the equation to the tangent is

$$\frac{y-\beta}{x-\alpha}=1-\frac{\alpha^2}{2\,!}+\frac{\alpha^4}{4\,!}-\ldots$$

But the series of the second member is equal to $\cos\alpha$. Hence for the required tangent we have finally the equation

$$y=\beta+[x-\alpha]\cos\alpha.$$

EXAMPLE 4. Similarly, the tangent to $y=\cos\alpha$ at the point $[\alpha, \beta]$ is

$$y=\beta-[x-\alpha]\sin\alpha.$$

EXAMPLE 5. What is the tangent to $y=\tan x$ at the point $[\alpha, \beta]$?

EXAMPLE 6, What is the tangent to $y=\log_e[1+x]$ at the point $[\alpha, \beta]$? We have

$$y=\log_e[1+x]=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots, \quad |x|<1;$$

and therefore [6] yields

$$\frac{y-\beta}{x-\alpha}=1-\alpha+\alpha^2-\alpha^3+\ldots=\frac{1}{1+\alpha}, \text{ since } |\alpha|<1.$$

Thus, for the required equation we have

$$y=[1+\alpha]x+\beta-\alpha^2.$$

EXAMPLE 7. Similarly, the tangent to $y=\log_e[1-x]$ at the point $[\alpha, \beta]$ is

$$y=[1-\alpha]x+\beta+\alpha^2.$$

[Remark. The chief difficulty in using the foregoing method with pupils beginning the study of Analytics will be in the discussion following equation $[3_a]$ above. A reference to Salmon (or Salmon-Fiedler, *l. c.*) will suffice to show that the difficulty is by no means insurmountable. If the tangent is defined as a straight line which passes through two consecutive points, the method by which Salmon shows that our $u^{(1)}=0$ is the tangent to our $u^{(1)}+u^{(2)}+\ldots+u^{(n)}=0$ will be easily within the grasp of the earnest student. Nothing else in the paper presents any intrinsic difficulty whatever. And the bright student will undoubtedly take interest in a simple general method and formula for solving numerous special problems which are continually arising.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

46. Proposed by T. W. PALMER, Professor of Mathematics, University of Alabama.

A borrows \$500 from a building and loan association and agrees to pay \$9.50 per month for 72 months, the first payment to be made at the end of the first month. What rate of interest does he pay? The association claims to charge only 8% (the legal rate in Alabama). How can the per cent. be figured out?

Solution of unsolved problem in Vol. II, p. 74, by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let r=rate per month, $12r$=rate per annum, p=sum borrowed, n= number of payments, q=cash payment. Then, from Algebra, we get

$$q=\frac{pr(1+r)^n}{(1+r)^n-1},\quad q=9\tfrac{1}{2},\quad p=\$500,\quad n=72.$$

$\therefore (q-pr)(1+r)^n=q$, and $(19-1000r)(1+r)^{72}=19$.
$\therefore r=.00911$, and $12r=.10932=10.932\%$ per annum.

ALGEBRA.

293. Proposed by C. E. WHITE, Vanderbilt University, Nashville, Tenn.

Prove by mathematical induction that $\dfrac{(x-a)^{m-1}}{(m-1)!}f^{m-1}(a)+\dfrac{(x-a)^{m-2}}{(m-2)!}+\dots+\dfrac{(x-a)^2}{2!}f''(a)+(x-a)f'(a)+f(a)$ will be the remainder when $f(x)$ is divided by $(x-a)^m$.

Solution by the PROPOSER.

If $\phi(x)$ be the quotient found by dividing $f(x)$ by $(x-a)$ we can write the identity

$$\frac{f(x)}{x-a}=\phi(x)+\frac{f(a)}{x-a}.$$

Differentiating both members and solving for $f(x)/(x-a)^2$,

$$\frac{f(x)}{(x-a)^2}=\frac{f'(x)}{x-a}+\frac{f(a)}{(x-a)^2}-\phi'(x). \tag{1}$$

Since $\phi(x)$ is integral,

$$F\left[\frac{f(x)}{(x-a)^2}\right]=F\left[\frac{f(x)}{x-a}\right]+F\left[\frac{f(a)}{(x-a)^2}\right], \tag{2}$$

where $F[\ ]$ represents the fractional part of the quotient found by performing the division indicated within the brackets.

When the two fractions in the second member of (2) are added, the numerator of the resulting fraction equals the numerator of the fraction in the first member.

Hence,

$$R\left[\frac{f(x)}{(x-a)^2}\right]=(x-a)R\left[\frac{f'(x)}{x-a}\right]+R\left[\frac{f'(a)}{(x-a)^2}\right],$$

where $R[\ ]$ represents the fractional part of the remainder found by performing the division indicated within the brackets.

$$\therefore R\left[\frac{f(x)}{(x-a)^2}\right]=(x-a)f'(a)+f(a). \tag{a}$$

Differentiating (1) and solving for $\dfrac{f(x)}{(x-a)^3}$,

$$\frac{f(x)}{(x-a)^3}=-\frac{f''(x)}{2!(x-a)}+\frac{f'(x)}{(x-a)^2}+\frac{f(a)}{(x-a)^3}+\frac{\phi''(x)}{2!}. \tag{3}$$

$$\therefore F\left[\frac{f(x)}{(x-a)^2}\right]=-F\left[\frac{f''(x)}{2!(x-a)}\right]+F\left[\frac{f'(x)}{(x-a)^2}\right]+F\left[\frac{f(a)}{(x-a)^3}\right]$$

and $R\left[\dfrac{f(x)}{(x-a)^3}\right]=-\dfrac{(x-a)^2}{2!}R\left[\dfrac{f''(x)}{x-a}\right]+(x-a)R\left[\dfrac{f'(x)}{(x-a)^2}\right]+R\left[\dfrac{f(a)}{(x-a)^3}\right].$

Since $R\left[\dfrac{f''(x)}{x-a}\right]=f''(a)$, $R\left[\dfrac{f(a)}{(x-a)^3}\right]=f(a)$, and $R\left[\dfrac{f'(x)}{(x-a)^2}\right]$

$$=(x-a)f''(a)f'(a),$$

$$R\left[\frac{f(x)}{(x-a)^3}\right]=\frac{(x-a)^2}{2!}f''(a)+(x-a)f'(a)+f(a). \tag{b}$$

Equations (a) and (b) prove the theorem true when $m=2$ or 3. We will now assume it true for $m\leq r$ and prove it true for $m=r+1$, or that

$$R\left[\frac{f(x)}{(x-a)^{r+1}}\right]=\frac{1}{r!}(x-a)^r f^r(a)+R\left[\frac{f(x)}{(x-a)^r}\right].$$

By mathematical induction,

$$\frac{f(x)}{(x-a)^m}=\frac{(-1)^m f^{m-1}(x)}{(m-1)!(x-a)}+\frac{(-1)^{m-1}f^{m-2}(x)}{(m-2)!(x-a)^2}+\cdots$$

$$+\frac{f'(x)}{(x-a)^{m-1}}+\frac{f(a)}{(x-a)^m}+\frac{(-1)^{m-1}\phi^{m-1}(x)}{(m-1)!}. \qquad (4)$$

If we let $m=r$ and $m=r+1$ in (4), we derive

$$R\left[\frac{f(x)}{(x-a)^r}\right]=R\left[\frac{f(a)}{(x-a)^r}\right]+(x-a)R\left[\frac{f'(x)}{(x-a)^{r-1}}\right]-\frac{(x-a)^2}{2!}R\left[\frac{f''(x)}{(x-a)^{r-1}}\right]+$$

$$\ldots(-1)^r\frac{(x-a)^{r-1}}{(r-1)!}R\left[\frac{f^{r-1}(x)}{x-a}\right], \qquad (5)$$

and $R\left[\frac{f(x)}{(x-a)^{r+1}}\right]=R\left[\frac{f(a)}{(x-a)^{r+1}}\right]+(x-a)\,R\left[\frac{f'(x)}{(x-a)^r}\right]$

$$-\frac{(x-a)^2}{2!}R\left[\frac{f''(x)}{(x-a)^{r-1}}\right]+\ldots(-1)^{r+1}\frac{(x-a)^r}{r!}R\left[\frac{f^r(x)}{x-a}\right]. \qquad (6)$$

The second member of (6) can be derived from the second member of (5) by dividing the fraction in the bracket of each term by $(x-a)$ and adding the term $\frac{(-1)^{r+1}(x-a)^r}{r!}R\left[\frac{f^r(x)}{x-a}\right]$.

Corresponding to the term $(-1)^{s-1}\frac{(x-a)^s}{s!}R\left[\frac{f^s(x)}{(x-a)^{r-s}}\right]$ in (5), we have the term $(-1)^{s-1}\frac{(x-a)^s}{s!}R\left[\frac{f^s(x)}{(x-a)^{r-s+1}}\right]$ in (6).

Since the value of m in each bracket of the second member of (6) is less than $r+1$, and since for any value of $m\leq r$ the remainder can be found from the remainder for the preceding value of m by adding a term of degree one less than the value of m, therefore,

$$(-1)^{s-1}\frac{(x-a)^s}{s!}R\left[\frac{f^s(x)}{(x-a)^{r-s+1}}\right]=(-1)^{s-1}\frac{(x-a)^s}{s!}\left(\frac{(x-a)^{r-s}}{(r-s)!}f^{r-s+1}(a)\right.$$

$$\left.+R\left[\frac{f^s(x)}{(x-a)^{r-s}}\right]\right)=\frac{(-1)^{s-1}}{s!}\frac{(x-a)^r}{(r-s)!}f^r(a)+(-1)^{s-1}\frac{(x-a)^s}{s!}R\left[\frac{f^s(x)}{(x-a)^{r-s}}\right].$$

In (6), the term $\frac{(-1)^{r-1}(x-a)^r}{r!} R\left[\frac{f^r(x)}{x-a}\right] = (-1)^{r-1}\frac{(x-a)^r}{r!}f^r(a)$.

Hence, $R\left[\frac{f(x)}{(x-a)^{r+1}}\right] = (-1)^{r-1}\frac{(x-a)^r}{r!}f^r(a) + \sum_{s=1}^{s=r-1}(-1)^{s-1}\frac{(x-a)^r}{s!\,(r-s)!}f^r(a)$

$$+\sum_{s=1}^{s=r-1}(-1)^{s-1}\frac{(x-a)^s}{s!}R\left[\frac{f^s(x)}{(x-a)^{r-s}}\right] = (-1)^{r-1}\frac{(x-a)^r}{r!}f^r(a)$$

$$+\sum_{s=1}^{s=r-1}(-1)^{s-1}\frac{(x-a)^r}{s!\,(r-s)!}f^r(a) + R\left[\frac{f(x)}{(x-a)^r}\right].$$

It remains to be proved that

$$(-1)^{r-1}\frac{(x-a)^r}{r!}f^r(a) + \sum_{s=1}^{s=r-1}\frac{(-1)^{s-1}(x-a)^r}{s!\,(r-s)!}f^r(a) = \frac{(x-a)^r}{r!}f^r(a)$$

$$\text{or that } \frac{(-1)^{r-1}}{r!} + \sum_{s=1}^{s=r-1}\frac{(-1)^{s-1}}{s!(r-s)!} = \frac{1}{r!}. \tag{7}$$

The factorial series $\sum_{s=1}^{s=r-1}\frac{(-1)^{s-1}}{s!\,(r-s)!} = \frac{1}{(r-1)!} - \frac{1}{2!(r-2)!} + \frac{1}{3!(r-3)!} -$

$$\cdots\frac{(-1)^{r-2}}{(r-2)!2!} + \frac{(-1)^r}{(r-1)!}. \tag{8}$$

Multiplying (8) by $r!/r!$ and adding and subtracting $1/r!$, the series

$$= \frac{1}{r!} + \frac{1}{r!}\left[(r-1) - \frac{r(r-1)}{2!} + \frac{r(r-1)\,(r-2)}{3!} + \ldots\ (r-1)\ \text{terms}\right].$$

The sum of the terms in the bracket $=(-1)^r\frac{(r-1)!}{(r-1)!} = +1$ if r be even and -1 if r be odd.

Hence, (7) is true for all integral values of r.

Since we have proved the theorem true for $m=2$ and 3, it must be true for $m=4$, 5, 6, and for any integral value of m.

It will be observed that the limiting value of the remainder is Taylor's expansion of the function.

Also solved by H. V. Spunar and G. B. M. Zerr.

294. Proposed by O. L. CALLECOT, Gettysburg, S. Dak.

Find the limit of $\displaystyle\sum_{n=1}^{n=\infty}\frac{2(n^2+3n+3)}{n(n+1)(n+2)(n+3)}$.

Solution by H. V. SPUNAR, C. E., East Pittsburg, Pa., and J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

$$\sum_{n=1}^{n=n}\frac{2(n^2+3n+3)}{n(n+1)(n+2)(n+3)}=\sum_{n=1}^{n=n}\left[\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\frac{1}{n+3}\right]$$

$$=\sum\frac{1}{n}-\sum\frac{1}{n+1}+\sum\frac{1}{n+2}-\sum\frac{1}{n+3}$$

$$=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots.+\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots.$$

$$-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\ldots.-\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}-\ldots.$$

$$+\frac{1}{3}+\frac{1}{4}+\ldots.+\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots.$$

$$-\frac{1}{4}-\ldots.-\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}-\ldots.$$

$$=1+\tfrac{1}{3}=1\tfrac{1}{3}.$$

Also solved by C. E. White and G. B. M. Zerr.

In order for the solution of this problem to be rigorous, the matter of convergency must be investigated. The equality

$$\sum_{n=1}^{n=n}\left[\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\frac{1}{n+3}\right]=\sum_{n=1}^{n=\infty}\frac{1}{n}-\sum_{n=1}^{n=\infty}\frac{1}{n-1}+\sum_{n=1}^{n=\infty}\frac{1}{n+2}-\sum_{n=1}^{n=\infty}\frac{1}{n+3}$$

assumed to hold in the above solution is not always true.

GEOMETRY.

328. Proposed by CHARLES GILPIN, JR., Philadelphia, Pa.

A sphere with the radius R is divided into two segments by a plane passed through it half way between the center and circumference. The smaller segment is divided into two parts by a plane passed through it at right angles to the base and cutting it half way between its center and circumference. Required the contents of the two parts of the segment.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa., and BENJAMIN F. FINKEL, Ph. D., Drury College, Springfield, Mo.

The equation of the sphere is $x^2+y^2+z^2=R^2$. The volume of $C-HIK$ is

329. Proposed by JOHN JAMES QUINN, Ph. D., Scottdale, Pa.

1. Determine the equation of the locus of a fixed point in a circle of radius r, rolling along the axis of an upright cylinder of the same radius, while the axis revolves (carrying the circle with it) through an angle equal to the central angle of the rolling circle formed by the radii to the fixed point and the point of contact.

2. Suppose the point projected into the surface of the cylinder.

3. What is the surface generated by the radius of the rolling circle?

4. What is the surface generated by a radius of the cylinder through the moving point?

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let a=the distance between the axis of the cylinder and the line about which it revolves. Also suppose the plane of the circle is determined by the line a and the axis of the cylinder. Let the circle, in its initial position, be in the xz plane with the point of contact in the same plane.

The position of the moving point at any time, using the axis of the cylinder as initial line and initial point of contact as origin, is $r\theta-r\sin\theta$, and $r(1-\cos\theta)$. We use the double sign, plus when the circle is not between the z-axis and the axis of the cylinder, and the minus when it is.

1. The locus is determined by $x=[a\pm r(1-\cos\theta)]\cos\theta$, $y=[a\pm r(1-\cos\theta)]\sin\theta$, $z=r\theta-r\sin\theta$, θ=central angle of circle.

2. If point is projected in surface of cylinder in plane of circle, the locus is determined by $x=(a\pm r)\cos\theta$, $y=(a\pm r)\sin\theta$, $z=r(\theta-\sin\theta)$.

3. Coordinates of the center of gravity of the moving radius of the circle are, $x=[a\pm r(1-\frac{1}{2}\cos\theta)]\cos\theta$, $y=[a\pm r(1-\frac{1}{2}\cos\theta)]\sin\theta$, $z=r(\theta-\frac{1}{2}\sin\theta)$.

$\therefore$ Surface$=rS$ where S is given as follows:

$$S=\int_0^\theta [4a^2\pm 8ar+8r^3\mp 4ar\cos\theta(4\sin^2\theta-1)+r^2(1-16\cos\theta)(1+\cos^2\theta)]^{\frac{1}{2}}]d\theta.$$

4. Coordinates of center of gravity of radius of cylinder through moving point are $x=(a\pm\frac{1}{2}r)\cos\theta$, $y=(a\pm\frac{1}{2}r)\sin\theta$, $z=r(\theta-\sin\theta)$.

$\therefore$ Surface$=rS$ where

$$S=\int_0^\theta [(a\pm\tfrac{1}{2}r)^2+r^2(1-\cos\theta)^2]^{\frac{1}{2}}d\theta.$$

Also solved by H. V. Spunar.

CALCULUS.

257. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

If $A=\int_0^\infty \dfrac{dx}{\sqrt{x}\,(2a+x)^n}$, $B=\int_0^\infty \dfrac{y^n dy}{\sqrt{y}\,(a^2+y^2)^n}$, find A/B.

Solution by J. SCHEFFER, A. M., Hagerstown, Md., and the PROPOSER.

Let $x=2a\tan^2\theta$, $y=a\tan\phi$, $m=\frac{1}{2}(2n-1)$.

$$\therefore A=\frac{2}{(2a)^{n-\frac{1}{2}}}\int_0^{\frac{1}{2}\pi}\cos^{2(n-1)}\theta\,d\,\theta=\frac{1}{2(2a)^m}\int_0^{\frac{1}{2}\pi}\cos^{2m-1}\theta\,d\,\theta=\frac{\sqrt{(\pi)}\Gamma(m)}{(2a)^m\Gamma\left(\frac{2m+1}{2}\right)}.$$

$$B=\frac{1}{a^{n-\frac{1}{2}}}\int_0^{\frac{1}{2}\pi}\tan^{n-\frac{1}{2}}\phi\cos^{2(n-1)}\phi\,d\phi=\frac{1}{a^m}\int_0^{\frac{1}{2}\pi}\sin^m\phi\cos^{m-1}\phi\,d\,\phi$$

$$=\frac{\Gamma\left(\frac{m+1}{2}\right)\,\Gamma\left(\frac{m}{2}\right)}{2a^m\,\Gamma\left(\frac{2\,m+1}{2}\right)}.$$

But $\Gamma\left(\frac{m+1}{2}\right)\,\Gamma\left(\frac{m}{2}\right)=\frac{\sqrt{(\pi)}\Gamma(m)}{2^{m-1}}$.

$$\therefore B=\frac{\sqrt{(\pi)}\Gamma(m)}{(2a)^m\Gamma\left(\frac{2\,m+1}{2}\right)}=A.\quad\text{Hence } A/B=1.$$

Exhaustive solutions, though differing in results from the one given here, were received from M. V. Spunar, Francis Rust, and T. G. Wodo.

258. Proposed by A. H. HOLMES, Brunswick, Maine.

Evaluate $\displaystyle\int_0^{\frac{1}{2}a}\frac{dx}{\sqrt{[2ax-x^2\sqrt{(a^2-x^2)}]}}$.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let $x=x\sin\theta$. Then

$$I=\int_0^{\frac{1}{2}a}\frac{dx}{\sqrt{[2ax-x^2\sqrt{(a^2-x^2)}]}}=\int_0^{\frac{1}{6}\pi}\frac{\cos\theta\,d\,\theta}{\sqrt{[2\sin\theta-a\sin^2\theta\cos\theta]}}.$$

$$\therefore\ I=\frac{1}{\sqrt{2}}\int_0^{\frac{1}{6}\pi}\left[1+\frac{a}{4}\sin\theta\cos\theta+\frac{3a^2}{32}\sin^2\theta\cos^2\theta+\frac{5a^3}{128}\sin^3\theta\cos^3\theta\right.$$

$$\left.+\frac{35a^4}{2048}\sin^4\theta\cos^4\theta+\frac{64a^5}{8192}\sin^5\theta\cos^5\theta+\frac{231a^6}{65536}\sin^6\theta\cos^6\theta+\ldots\right]\frac{\cos\theta\,d\,\theta}{\sqrt{(\sin\theta)}}$$

$$=1+\frac{31}{15}\cdot\frac{a^2}{2^9}+\frac{85015}{1989}\cdot\frac{a^4}{2^{19}}+\frac{2350494}{38675}\cdot\frac{a^6}{2^{28}}+\ldots.+\frac{1}{\sqrt{2}}\int_0^{\frac{1}{6}\pi}\left[\frac{a}{4}\sin\theta\cos\theta\right.$$

$$+\frac{5a^3}{128}\sin^3\theta\cos^3\theta+\frac{63a^5}{8192}\sin^5\theta\cos^5\theta+\ldots\Bigg]\frac{\cos\theta\,d\theta}{\sqrt{(\sin\theta)}}.$$

$$\frac{1}{\sqrt{(\sin\theta)}}=(1-\cos^2\theta)^{-\frac{1}{4}}.$$

$$\therefore I=1+\frac{31}{15}\cdot\frac{a^2}{2^9}+\frac{85015}{1989}\cdot\frac{a^4}{2^{19}}+\frac{2350494}{38675}\cdot\frac{a^6}{2^{28}}+\ldots$$

$$+\frac{1}{\sqrt{2}}\left[\frac{a}{35.\,2^5}\left(\frac{341}{3}-\frac{20611\sqrt{3}}{2^9}+\ldots.\right)+\frac{a^3}{21.\,2^8}\left(\frac{41}{3}-\frac{2616\sqrt{3}}{2^{10}}+\ldots.\right)\right.$$

$$\left.+\frac{a^5}{11.\,2^{10}}\left(1-\frac{8181\sqrt{3}}{2^{14}}+\ldots.\right)\right]+\ldots.$$

a cannot be greater than $\frac{8}{3}\sqrt{3}$.

This solution, to be complete, should have investigated the matter of convergency and, since the function vanishes at the lower limit, also the condition of determinateness.

The proposer of 259 (256) suggests that the equation be changed to $(1+y+2axy)dx+x(1+x)dy=0$.

MECHANICS.

211. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

A smooth elliptic wire, axis vertical, has a small ring sliding on it, connected by elastic strings with each focus. Either string is just unstretched when the ring is nearest the corresponding focus. The modulus of elasticity is W/n, where Woz. is the weight of the ring. Find the distance of the ring from the upper focus in the different positions of equilibrium, and in each case discuss the nature of the equilibrium.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let F be the upper focus of the elliptic wire, semi-major axis a, eccentricity e. Let r, $2r-a$ be the lengths of the strings from the upper and lower foci to the ring at the point P on the curve; G the intersection of the normal from P with the axis major; θ the angles the strings make with the tangent at P; ϕ the angle the tangent at P makes with the major axis; T the tension of string r; Q the tension of string $2a-r$; m the weight of the ring and strings that cause a downward force due to gravity; $a(1-e)$=unstretched length of each string.

Then $PF=r$, $GF=er$, $\sin GPF=\cos\theta$, $\sin PGF=\cos\phi$, $\sin PGF:\sin GPF=r:er$. $\therefore\cos\phi/\cos\theta=1/e$.

Also $r=a(1-e)(1+Tn/W)$, $2a-r=a(1-e)(1+Qn/W)$.

$$\therefore\ T=\frac{W(r-a+ae)}{an(1-e)},\quad Q=\frac{W(a+ae-r)}{an(1-e)}.$$

I. For equilibrium, $T\cos\theta=Q\cos\theta+m\cos\phi$.

$$\therefore\ \frac{T-Q}{m}=\frac{\cos\phi}{\cos\theta}=\frac{1}{e}.\quad \therefore\frac{2\,W(r-a)}{amn(1-e)}=\frac{1}{e}.$$

$$\therefore\ r=a+\frac{amn(1-e)}{2eW}=a+\frac{an\,(1-e)}{2e},\ \text{when } m=W.$$

When the ring is displaced it will tend to regain this same position of equilibrium.

II. For equilibrium, $T\cos\theta=Q\cos\theta=0$.

$\therefore r=a(1-e)$ and $r=a(1+e)$, or the upper and lower vertices. When the ring is displaced from either of these positions it will tend to equilibrium in I.

212. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, Eng.

A peg A is vertically d feet above a peg B. A string AD, a feet long, with two equal, jointed rods DC, CB form the whole figure. Discuss the position of equilibrium.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let θ=the angle the string makes with the vertical; ϕ=the angle DC makes with the vertical; ψ=the angle CB makes with the vertical; b=the length of each rod; W its weight. Also regard the string as weightless, and let x=the depth of the center of gravity of the system below A.

$$\therefore x=[w(a\cos\theta+\tfrac{1}{2}b\cos\phi)+w(d\pm\tfrac{1}{2}b\cos\psi)]/2w\ldots(1).$$

Projecting vertically, we get $a\cos\theta+b\cos\phi\mp b\cos\psi=d\ldots(2)$.

$$\text{Also, } a^2+d^2-2ad\cos\theta=2b^2-2b^2\cos(\psi-\phi)\ldots(3).$$

$a\cos\theta$ from (2) in (1) and (3) gives $x=[w(2d-\frac{1}{2}b\cos\phi\pm\frac{3}{2}b\cos\psi)]/2w\ldots(4)$.

$$a^2-d^2+2db\cos\phi\mp2db\cos\psi=2b^2-2b^2\cos(\psi-\phi)\ldots(5).$$

Differentiating (4) and (5), we get $3\sin\psi\,d\psi=\pm\sin\phi\,d\phi\ldots(6)$.

$$[b\sin(\psi-\phi)\pm d\sin\psi]d\psi=[b\sin(\psi-\phi)-d\sin\phi]d\phi\ldots(7).$$

Eliminating dx and $d\phi$ between (6) and (7),

$$b\sin(\psi-\phi)(3\sin\psi\pm\sin\phi)=2d\sin\phi\sin\psi\ldots(8).$$

(5) and (8) determine the equilibrium. The $\pm$ sign is used as follows: if a is long enough to permit C to fall below B use the upper sign; if not, use the lower.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

148. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Find all the multiply perfect numbers of n different prime factors and of multiplicity $n-1$.

No solution received.

149. Proposed by REV. R. D. CARMICHAEL, Anniston, Ala.

Prove that every prime of the form $4n+1$ may be expressed as the sum of two parts r and s such that r^2+rs+s^2+1 is divisible by the prime.

I. Solution by E. B. ESCOTT, Ann Arbor, Mich.

Let $p=r+s=$a prime of the form $4n+1$. Substitute $s=p-r$ in $r^2+rs+s^2+1\equiv 0$ (mod. p) and we have $r^2+r(p-r)+(p-r)^2+1\equiv 0$ (mod. p) or $r^2+1\equiv 0$ (mod. p), which is always possible when p is a prime of the form $4n+1$, and in many other cases.

Example. If $p=5$, $r^2+1\equiv 0$ (mod. 5) has the solution $r\equiv 2$ (mod. 5). $\therefore r=2$, $s=3$.

II. Solution by E. T. BELL, University of Washington.

When $p\equiv 1$ (mod. 4) : $\left(\dfrac{-1}{p}\right)=1$ (Legendre's symbol).

Hence, for r a root of $x^2\equiv -1$ (mod. p), $rp-(r^2+1)\equiv 0$ (mod. p); whence, putting $s=p-r$, $rs-1\equiv 0$ (mod. p).

Therefore, $p^2-(rs-1)\equiv 0$ (mod. p); or, r^2+rs+s^2+1 is a multiple of p; which establishes the stated theorem.

III. Solution by W. F. KING, Ottawa, Canada.

Since $r+s$ is a prime of form $4n+1$, it may be resolved in one way into the sum of two squares. Suppose it to be so resolved, and to be equal to h^2+k^2.

By the usual process, express h/k as a continued fraction in the form

$$\frac{h}{k}=x+\frac{1}{x_1+}\,\frac{1}{x_2+}\,\frac{1}{x_3+}\cdots$$

Form the successive convergents, and let the convergent next before the last (h/k), be (a/b).

Then $ak-bh=\pm 1$.

And $(ah+bk)^2+(ak-bh)^2=a^2h^2+b^2k^2+a^2k^2+b^2h^2=(a^2+b^2)(h^2+k^2)$.

Therefore, $(ah+bk)^2+(ak-bh)^2$, or $(ah+bk)^2+1$, is divisible by h^2+k^2, or $r+s$.

Now $a<h$, $b<k$. $\therefore ah+bk<h^2+k^2$.

Calling $ah+bk$, r, we have r^2+1, and therefore $r^2+1+s(r+s)$, or r^2+rs+s^2+1, divisible by $r+s$.

It is evident that the number $ah+bk$ which is less than h^2+k^2 and is such that $(ah+bk)^2+1$ is divisible by h^2+k^2, can always be found.

Also solved by G. B. M. Zerr.

PROBLEMS FOR SOLUTION.

ALGEBRA.

303. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Evaluate the determinant

$$\begin{vmatrix} D_1 & x_1x_2 & x_1x_3 & \dots & x_1x_n \\ x_1x_2 & D_2 & x_2x_3 & \dots & x_2x_n \\ x_1x_3 & x_2x_3 & D_3 & \dots & x_3x_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1x_n & x_2x_n & x_3x_n & \dots & D_n \end{vmatrix}$$

304. Proposed by C. N. SCHMALL, New York City.

A policeman on a motor-cycle starts in pursuit of an automobile when the latter has a headway of $\frac{1}{2}$ a mile. A pedestrian who is $\frac{1}{4}$ of a mile ahead of the auto and who is walking at the rate of 5 miles an hour, notices that when the auto overtakes him the policeman is only $\frac{5}{12}$ of a mile behind the auto, and $2\frac{1}{2}$ miles from where the officer started; he overtakes the auto. How long did the chase last?

305. Proposed by S. A. COREY, Hiteman, Iowa.

Prove or disprove, that $\displaystyle\sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^2+4}=\frac{\pi}{4}-\frac{\pi}{8}\left(\frac{\cosh\pi}{\sinh\pi}\right)$.

GEOMETRY.

336. Proposed by F. H. HODGE, The University of Chicago.

A man owning a rectangular field $b=300$ feet by $a=600$ feet, wishes to lay out driveways of equal width having the diagonals of the field as center lines and such that the area of the driveways shall be n/m=one-half, of the area of the field. Determine the width of the driveways.

337. Proposed by T. N. HILDEBRANT, The University of Chicago.

Required the locus of the vertices of the parabolae having a given focus and passing through a given point.

338. Proposed by C. N. SCHMALL, 239 East 7th Street, New York.

Given the base and vertical angle of a triangle, find the locus of the center of its "nine-point" circle. [Ex. 28, p. 65, Casey's *Sequel to Euclid.*]

CALCULUS.

263. Proposed by V. M. SPUNAR, M. S., C. E., East Pittsburg, Pa.

Find a point such that the sum of the squares of its distances from n given points shall be a minimum, and prove that the value so found is $1/n$th part of the sum of the squares of the mutual distances between the n points, taken two and two.

264. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

The join of the center of curvature of a curve to the origin is at α to the initial line. Prove that with the usual notation:

$$\frac{d\alpha}{d\psi}\left[\left(\frac{dp}{d\psi}\right)^2+\left(\frac{d^2p}{d^2\psi}\right)^2\right]=\frac{dp}{d\psi}\cdot\frac{d\rho}{d\psi}.$$

MECHANICS.

219. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

A rod length $a\sqrt{3}$, weight W, has at each end a smooth ring which can slide on a vertical circle radius r. Each ring is attached by an elastic string (natural lengths a, b; moduli μa, μb) to the highest point of the circle. Find the inclination of the rod to the horizon in a position of equilibrium.

220. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Four particles A, B, C, D lie on a smooth table at the corners of a rhombus. AB, BC, CD, DA are light inextensible strings connecting the particles. The angle at A is acute. A blow is given to A along the diagonal, away from C. Find the ratio of the initial velocity of C to that of A.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

155. Proposed by PROF. R. D. CARMICHAEL, Anniston, Alabama.

If p and q are primes and m and n are any integers, find the cases in which the equation $p^m-q^n=1$ may be satisfied.

156. Proposed by A. H. HOLMES, Brunswick, Maine.

Find integral values for a, b, c, d, and e in the equation, $a^2+b^2+c^2+d^2=e^2$.

AVERAGE AND PROBABILITY.

199. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

A circle is inscribed in a given square. Two points are taken at random within the square but without the circle. What is the chance the distance between the points does not exceed the side of the square?

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ON DeMOIVRE'S QUINTIC.

By DR. R. L. BORGER, University of Illinois.

§1. For the domain of rational numbers, DeMoivre's quintic

(1) $$x^5+px^3+\tfrac{1}{5}p^2x+r=0,$$

for values of p and r making the discriminant

$$\triangle\equiv\left(\frac{r}{2}\right)^2+\left(\frac{p}{5}\right)^5\neq 0,$$

will be shown to have as its Galois group *either the metacyclic group* G_{20} *or a cyclic group* C_4. We may then readily deduce the following properties:

DeMoivre's quintic is solvable by radicals.

Either all the roots are real or only one root is real.

Not more than one root is rational; if the equation is reducible in $R(1)$, *its left member is the product of a linear and an irreducible quartic factor.*

If the equation is irreducible in $R(1)$, *any root is a rational function of an arbitrary pair of roots.*

To determine the Galois group of (1), we make use on the one hand of Cayley's resolvent sextic for any quintic, and on the other hand of the following lemma:*

If we know a rational function of the roots of an algebraic equation $f(x)=0$ having the properties:

(i) That it is formally invariant under the substitutions of a group G' and under no others.

(ii) That it has a value in the domain of rationality.

(iii) That it is distinct from its conjugates under the substitutions of the symmetric group $G_n!$, then the Galois group G of $f(x)=0$ is a subgroup of G'.

Dickson, *Algebraic Equations*, p. 59, §65.

§2. We exclude those values of p and r for which the discriminant $\triangle=0$. They give rise to equal roots and these may be removed by the process of highest common divisor. The function

$$\phi\equiv(x_1x_2+x_2x_3+x_3x_4+x_4x_5+x_5x_1)-(x_1x_3+x_2x_4+x_3x_5+x_4x_1+x_5x_2)$$

belongs to the group G_{10} consisting of the substitutions

$$\begin{array}{l} 1;\quad (12345);\quad (13524);\quad (14253);\quad (15432);\\ (12)(35);\quad (25)(34);\quad (15)(24);\quad (14)(32);\quad (13)(45). \end{array}$$

Under the substitutions of G_{60}, ϕ takes six values, which are the roots of a resolvent sextic. For the general quintic Cayley has computed* this resolvent sextic, which becomes for equation (1):

$$\phi^6-7p^2\phi^4+11p^4\phi^2-\frac{32\,(5^5r^2+2^2p^5)}{5^2\sqrt{5}}\phi+4000pr^2+\tfrac{3}{25}p^6=0. \tag{2}$$

One root of (2) is $\phi=p\sqrt{5}$. By differentiating (2) with respect to ϕ we see that this root is simple unless

$$121p^5+5^5r^2=0. \tag{3}$$

We now divide the discussion into the two cases
I) p and r not satisfying (3).
II) p and r satisfying (3).

I) In this case, ϕ is distinct from its conjugates under G_{60}. Hence $\phi^2=5p^2$ belongs to G_{20}† and is distinct from its conjugates. Hence (§1), G_{20} contains the Galois group of (1). The Galois group G for the domain $R(1)$ may then be

$$G_{20},\ G_{10},\ C_5,\ C_4,\ G_2,\ \text{or}\ G_1\equiv 1.$$

The groups G_{10}, G_5, G_2, G_1 may be at once excluded. By the definition of the Galois group of an equation, every rational function of the roots which remains invariant under the substitutions of G is rationally known. If G is G_{10} or a subgroup of it, then ϕ, belonging to G_{10}, would be rationally known. Since $\phi=p\sqrt{5}$ this is impossible unless $p=0$. Hence when $p\neq 0$, G is not contained in G_{10}.

*Cayley, *Collected Mathematical Papers*, Vol. IV, p. 319.

†The substitutions of G_{20} are given by $\binom{\alpha}{\alpha x+\beta}$, $\binom{\alpha=1\ 2\ 3\ 4}{\beta=0\ 1\ 2\ 3\ 4}$.

If $p=0$ we know that (1) reduces to a binomial equation p and its group is metacyclic when r is not the fifth power of a rational number. If $r=k^5$ (k rational) the group G is then C_4. Hence when p and r do not satisfy (3), $G=C_4$ or G_{20}.

§3. Next, we consider the case in which p and r satisfy (3). By solving (3) we find

$$r=\frac{11p^2}{5^2}\sqrt{\frac{p}{5}}. \tag{4}$$

Since r must be a rational number, $\sqrt{\frac{p}{5}}=a$ (a rational),

$$p=5a^2, \quad r=11a^5. \tag{5}$$

Substituting these values in (1), we get

$$x^5+5a^2x^3+5a^4x+11a^5=0. \tag{6}$$

This equation has the root $x=-a$, and the depressed equation is

$$x^4-ax^3+6a^2x^2-6a^3x+11a^4=0. \tag{7}$$

Calling the roots of (7) x_1, x_2, x_3, x_5, and setting

$$y_1=x_1x_2+x_3x_5,$$
$$y_2=x_1x_3+x_2x_5,$$
$$y_3=x_1x_5+x_2x_3,$$

we obtain the cubic resolvent of (7),

$$y^3-6a^2y^2-38a^4y+217a^6=0. \tag{8}$$

The roots of (8) are:

$$y_1=7a^2;\ y_2=\frac{a^2}{2}(-1+5^2\sqrt{5});\ y_3=\frac{a^2}{2}(-1-5^2\sqrt{5}); \tag{9}$$

$y_1=x_1x_2+x_3x_5$ belongs to the group

$G_8\equiv[1, (12); (35); (1325); (1523); (12)(35); (13)(25); (15)(23)]$.

And since y_1 is distinct from its conjugates under G_{24} the Galois group of

(7) is G_8 or a subgroup of G_8. As y_2 and y_3 are irrational the group for the domain $R(1)$ cannot be contained in

$$G_4 \equiv [1;\ (12)(35);\ (13)(25);\ (15)(23)].$$

The function $\psi \equiv (x_1+x_2)-(x_3+x_5)=a\sqrt{5}$ belongs to the group $H_4 \equiv [1;\ (12);\ (35);\ (12)(35)]$. Since the value of ψ is irrational* G is not contained in H_4.

The function $\chi \equiv (x_1-x_2)(x_3-x_5)[x_1+x_2-(x_3+x_5)] \equiv a^3 5^5$ belongs to the group $C_4 \equiv [1;\ (1325),\ (1523);\ (12)(35)]$, since χ is rational and takes two values under the substitutions of G_8.

Therefore, $G=C_4$ or one of its subgroups. G cannot be a subgroup of C_4 as the subgroups of C_4 are contained in H_4. Therefore, $G=C_4$.

Hence when p and r satisfy (3), the group of (1) is C_4.

§4. We have now proved that *the Galois group of DeMoivre's quintic for the domain $R(1)$ is either the cyclic group C_4 or the metacyclic group G_{20}*. We therefore get the following results:

I. *DeMoivre's quintic is solvable by radicals.* The solution can be effected by the well known substitution $x=y-p/5y$.

Since the group may be C_4 the equation may be reducible. Hence

II. *If the equation is reducible it must reduce to the product of a linear factor and an irreducible quartic factor.*

As an equivalent form of II, we have,

III. *DeMoivre's quintic can never have more than one rational root.*

By means of a property of metacyclic equations† we may also conclude that

IV. *All the roots of DeMoivre's quintic are real or only one of them is real.*

If the group of the equation is G_{20} the equation is metacyclic‡ and,

V. *Each root is a rational function of an arbitrary pair of them.*

This problem was suggested to me by Prof. L. E. Dickson and I wish to thank him for criticisms and suggestions in connection with its solution.

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*a is not equal to 0 because of (3) and not both p and $r=0$.

†Weber, *Algebra*, I, p. 620, VIII.

‡Weber, *Algebra*, I, p. 618, VI.

ON PLANE ALGEBRAIC CURVES SYMMETRICAL WITH RESPECT TO EACH OF TWO RECTANGULAR AXES.*

By PROFESSOR R. D. CARMICHAEL, Anniston, Alabama.

The object of this paper is to point out the form of the Cartesian equation of plane algebraic curves symmetrical with respect to each of two rectangular axes, and to classify such curves of the fourth degree—the grouping in classes being determined by certain geometric properties common to those of each class.

The axes of symmetry will be taken as the axes of coordinates. Then if one point is (α, β), three other points are evidently $(-\alpha, \beta)$, $(-\alpha, -\beta)$, $(\alpha, -\beta)$. Since to each value α of x there correspond two values $+\beta$ and $-\beta$ of y, it follows that y enters into the equation only in even powers. In the same manner it may be shown that x enters only in even powers. Therefore,

If a plane algebraic curve symmetrical with respect to each of two rectangular axes is referred to these axes as axes of coordinates, its equation has only terms of even degree in both x and y. The curve is, therefore, of even order. These conditions are evidently sufficient, as well as necessary, for the existence of the defined symmetry; for if x and y enter to only even degrees, to each point (α, β) of the locus will correspond three others $(-\alpha, \beta)$, $(-\alpha, -\beta)$, $(\alpha, -\beta)$ — a condition which is clearly sufficient for the existence of the symmetry in consideration.

This result indicates, as it should, that the circle, ellipse, and hyperbola all possess such symmetry while the parabola does not.

The classification of quartic curves possessing such symmetry is not so simple a matter. For the resolution of this question will be required certain of Plücker's equations. By n, m, δ, τ, ρ, ι, we shall as usual represent respectively, the order, class, number of double points, number of double tangents, number of cusps, number of points of inflection of the curve. The Plücker equations which we shall require are then the following:

$$m=n(n-1)-(2\delta+3\rho), \tag{1}$$
$$n=m(m-1)-(2\tau+3\iota), \tag{2}$$
$$\iota=3n(n-2)-(6\delta+8\rho), \tag{3}$$
$$\rho=3m(m-2)-(6\tau+8\iota). \tag{4}$$

We have now to find the cases in which these equations can be satisfied subject to the condition that the defined symmetry exists.

Evidently, the singularities which are not on the axes can enter only by fours; and to each singularity on one axis and not at the origin must correspond another on the same axis and on the opposite side of the origin.

*Read before the Chicago Section of the American Mathematical Society, April 18, 1908.

Hence, singularities on an axis and not at the origin must enter by pairs. Moreover, it is easy to see that cusps and points of inflection can enter even at the origin only in pairs. Again, if a point of inflection occurs anywhere on an axis there will be two coincident points of inflection, as may be seen by a consideration of the geometric nature of a point of inflection and of the curves having the defined symmetry. To each of these two will correspond another on the opposite side of the origin. Hence, points of inflection not at the origin can enter only by fours. Again: if a double point occurs on an axis, there will be two coincident double points unless the two branches meeting in the point osculate and have the axis as a common tangent. If such an osculating point exists on one side of the origin it exists also on the axis on the other side. Therefore, since each point of tangency counts as two points of intersection, the axis will intersect the curve in eight points —a condition which is impossible for quartic and sextic curves. Therefore, in quartic and sextic curves double points not at the origin enter only by fours. In the same way it may be shown that in these quartic and sextic curves cusps not at the origin can enter only by fours.

Let us now apply the results of the foregoing paragraph to quartic curves having the defined symmetry. We have seen that ρ and ι are even. But by (3), $\rho \not> 3$ and $\iota \not> 24$. Hence $\rho=0$ or 2. How, if $\rho=2$, both cusps must be at the origin; for we have seen that cusps not at the origin can enter only by fours. If two cusps exist at the same point there will be a double point. But with $\rho=2$ and δ not zero, equations (1) to (4) can exist at the same time only when $\rho=2$, $\delta=1$, $\tau=1$, $\iota=2$. In the present case the two points of inflection can exist only at the origin; for such points not at the origin can enter only by fours. This introduces a second double point at the origin contrary to the equation $\delta=1$. Hence $\rho \neq 2$. Then $\rho=0$. With $\rho=0$, only the following sets of values will satisfy equations (1) to (4):

	I.	II.	III.	IV.
$\rho=$	0,	0,	0,	0.
$\delta=$	3,	2,	1,	0.
$\tau=$	4,	8,	16,	28.
$\iota=$	6,	12,	18,	24.
$m=$	6,	8,	10,	12.

These are the only four possible cases for quartic curves. Moreover, since double points not at the origin can enter only by fours, it follows that in each of the first three cases, the curves must pass through the origin; and therefore for these cases the independent term in the equation is always lacking.

It will be observed that in order to determine to which class any given curve belongs we have only to inquire how many double points it has at the origin. The first case, $\delta=3$, can exist only when the origin alone is the locus

of the equation; for otherwise some line through the origin would cut the curve in five points, which is impossible.

We may also apply certain of the foregoing results to sextic curves whose equation is of the form

$$a_0x^6+a_1x^4y^2+a_2x^2y^4+a_3y^6+a_4x^4+a_5x^2y^2$$
$$+a_6y^4+a_7x^2+a_8y^2+a_9=0,\quad a_9\neq 0.$$

Since this curve does not pass through the origin, its double points, points of inflection, and cusps can enter only by fours. Then from (1) it follows that $\rho=0$, 4, or 8; and $\delta=0$, 4, 8, or 12. Also, from (2) $m\nless 3$. Therefore, equations (1) to (4) yield only the following sets of values:

	I.	II.	III.	IV.	V.	VI.	VII.
$\rho=$	0,	0,	0,	0,	4,	4,	8.
$\delta=$	0,	4,	8,	12,	0,	4,	0.
$\iota=$	72,	48,	24,	0,	40,	16,	8.
$\tau=$	324,	156,	52,	12,	40,	36,	0.
$m=$	30,	22,	14,	6,	18,	10,	6.

ON THE DETERMINATION OF CONICS THROUGH TWO POINTS, THE MAJOR AXIS AND ONE FOCUS BEING GIVEN.

By T. H. HILDEBRANDT, The University of Chicago.

In the consideration of the question of the determination of constants in the application of the Principle of Least Action to Planetary Motion, we need to solve the following problem:

"Given two points, a focus, and a major axis, of a conic, besides its nature (*i. e.*, ellipse, hyperbola, or parabola). Required the number of conics that will satisfy the given conditions." This problem has been solved for the case of the ellipse by Jacobi.* The regions in which the second point must lie in order that we obtain a real solution, have also been determined.† So far as I know, however, the problem has not been considered for the cases in which the conic is an hyperbola or a parabola.

I. *The Hyperbola.* We need to use the following property of the hyperbola: The difference of the lengths of the focal radii of an hyperbola is equal in length to the major axis.

Let S be one focus and F another, and let P be a point on the hyper-

*Jacobi, *Vorlesungen ueber die Dynamik.* Werke, Suppl., p. 48.
†Todhunter, *Researches in Calculus of Variations*, p. 162.

bola with distance $SP=r$. Let $2a$ be the length of the major axis. Then if P lies on the branch of the hyperbola adjacent S, we have

$$PF=r+2a.$$

If P lies on the branch adjacent F, then

$$PF=r-2a.$$

Suppose S is the given focus, P and Q the given points. Let $PS=r_0$, $QS=r_1$, and let $2a$ be the length of the major axis. Denote by F the second focus of the hyperbola. Then we must distinguish between the following four possibilities:

I. P and Q both on the branch of the hyperbola adjacent S.

II. P and Q both on the branch of the hyperbola adjacent F.

III. P on the branch adjacent S and Q on the branch adjacent F.

IV. P on the branch adjacent F and Q on the branch adjacent S.

Case I. (See Fig. I) P and Q on the Branch Adjacent S.

From the above it is evident that the second foci of the hyperbolae through P will lie on the circle of radius $2a+r_0$ with P as center. The second foci of the hyperbolae through Q will lie on the circle of radius $2a+r_1$ with Q as center. Hence the second foci of the hyperbolae through P and Q will be the intersections of these two circles. There will always be two real intersections, but they may be identical. Let us consider when they coincide, that is, when the circles are tangent to each other. Evidently the circles will be tangent internally. Hence the distance PQ will be the difference of their radii:

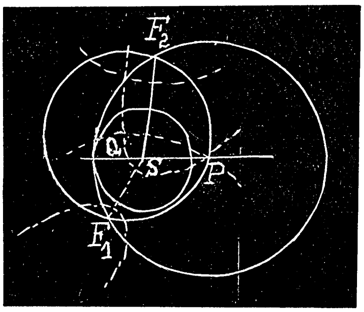


Fig. I.

$$r_0-r_1 \text{ or } r_1-r_0 \text{ according as } r_1<>r_0.$$

The locus of the points Q for which this happens is then the half ray SP. Evidently when Q lies on this line the hyperbolae degenerate into the straight line SPQ. Since r_0-r_1 is the smallest distance possible between two points P and Q we see that there will be no points Q for which the two circles will not intersect. We have then: through any two points P and Q it is possible to pass two hyperbolae P and Q lying on the branch adjacent S, provided P, Q, and S do not lie on the same straight line, with P and Q on the same side of S.

Case II. (See Fig. II) P and Q on the Branch Adjacent F.

In this case the second foci are the intersections of the circle of radius r_0-2a and center P with the circle of radius r_1-2a and center Q. Evidently we must have $r_0>2a$ and $r_1>2a$, *i. e.*, P and Q lying outside the circle of radius $2a$ with S as center. Moreover, with P fixed it is easily evident that there are points Q for which the circles do not intersect or are tangent Let us find the locus of the points Q for which they are tangent. The circles will be tangent externally, and so we have

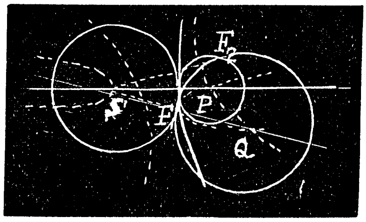


Fig. II.

$$PQ=(r_0-2a)+(r_1-2a)=r_0+r_1-4a.$$

Moreover, $QS=r_1$. Hence, $PQ-QS=r_0-4a$ or $QS-PQ=4a-r_0$, according as $r_0><4a$.

Therefore these points Q satisfying the condition $PQ-QS=r_0-4a=$a constant, trace a branch of an hyperbola with foci S and P. This hyperbola passes through O, the point of intersection of SP with the circle around P. If $r_0<4a$, the branch will be adjacent to P, and if $r_0>4a$ it will be adjacent S. In either case, if Q lies to the right of this branch, there are no real hyperbolae satisfying the given conditions. If Q lies to the left, there are two real hyperbolae and when Q lies on the branch of the hyperbola, the two required hyperbolae coincide.

Cases III and IV are really not of interest from the standpoint of the Calculus of Variations. We discuss them to make the discussion complete and on account of their intrinsic interest.

Case III. (See Fig. III) P on the Branch Adjacent S and Q on the Branch Adjacent F.

The second foci are the intersections of the circle of radius $2a+r_0$, center P, with the circle of radius r_1-2a, center Q. We must evidently have $r_1>2a$ in order to obtain real solutions. Moreover, there exist points Q for which the two circles do not intersect or are tangent. Let us find the locus of the points Q for which they are tangent. They will be tangent externally. Hence

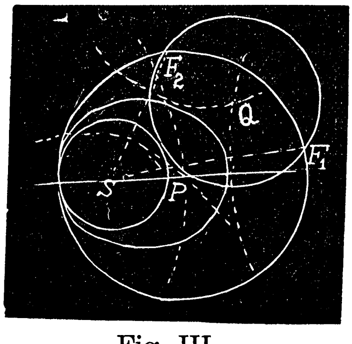


Fig. III.

$$PQ=2a+r_0-(r_1-2a)=4a+r_0-r_1,\quad QS=r_1,$$

and therefore $PQ+QS=4a+r_0$; *i. e.*, these points Q trace an ellipse with foci

P and S and major axis $4a+r_0$. For points Q within this ellipse no real solutions are possible; for points without, we have two real hyperbolae, and for points on the curve, the hyperbolae are coincident.

Case IV. (See Fig. IV) P on the Branch Adjacent F and Q on the Branch Adjacent S.

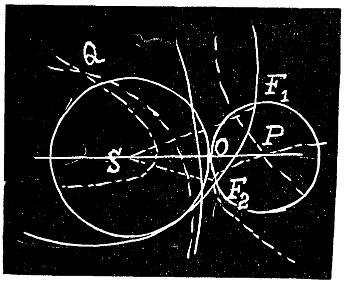


Fig. IV.

In this case the second foci of the hyperbolae through P and Q are the intersections of the circle of radius r_0-2a, center P, with the circle of radius r_1+2a, center Q. It is easily seen that there are points Q for which the two circles do not intersect or are tangent. We find the locus of the latter points. The circles will evidently be tangent internally. Hence,

$$PQ=2a+r_1-(r_0-2a)=4a+r_1-r_0, \quad QS=r_1,$$

and therefore $PQ-QS=4a-r_0$ or $QS-PQ=r_0-4a$, according as $r_0><4a$.

Hence the points Q trace a branch of an hyperbola with P and S as foci and major axis $4a-r_0$ or r_0-4a. This is the second branch of the hyperbola of Case II. It will be adjacent S if $r_0<4a$ and adjacent P if $r_0>4a$. For all points Q lying to the left of this curve there are no real hyperbolae satisfying the given conditions; for points to the right there are two; and for points on the curve, the hyperbolae become coincident.

Figs. V and VI show a comparison of the regions within which Q must lie in order that it be possible to draw real hyperbolae through P and Q satisfying the respective conditions.

It is easily evident that the ellipse of Case IV is the envelope of the hyperbolae through a point P lying on the branch of the hyperbolae adjacent the given focus S. The hyperbola of Cases II and IV is the envelope of the hyperbolae through a point P lying on the branch of the hyperbolae remote from the focus S.

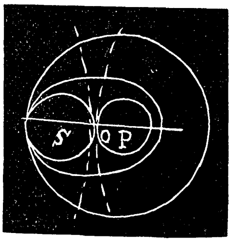


Fig. V.

Case II. Points to the right of - - - -
Case III. Points outside ellipse.
Case IV. Points to the left of — - — - —

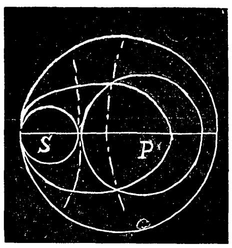


Fig. VI.

Case II. Points to the right of - - - -
Case III. Points outside ellipse.
Case IV. Points to the left of — - — - —

Parabolae. (See Fig. VII). In order to find the parabolae through two given points P and Q, having a given focus S, we need to find the directrices. The directrices of the parabolae through P are evidently the tangents to the circle of radius SP and center P, and those of the parabolae through Q, the tangents to the circle of radius SQ and center Q. Hence the parabolae through P and Q have the common tangents of these two circles as directrices. These tangents will always be real, and distinct, unless P, Q, and S lie on the same straight line and P and Q lie on the same side of S. Then there is only one common tangent, one that passes through S, and hence in this case the parabolae become coincident and degenerate into the straight line PQS. There is only one other exceptional case, when P, Q, and S lie on the same line and P and Q lie on opposite sides of S. Then there will be three common tangents, and there will consequently be three real parabolae through P and Q. One of these is the straight line PQS.

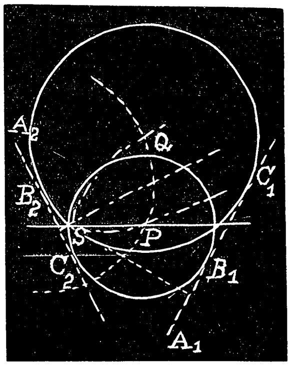


Fig. VII.

The parabolae through a point P with focus S have no envelope.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

295. Proposed by CHARLES GILPIN, JR., Philadelphia, Pa.

In the equation $x^3-ax\pm b=0$, we have the following relation between the coefficients and the roots: (1) When $a^3/b^2=6.75$ there are three real roots, two of which are equal; (2) when $a^3/b^2<6.75$ there are two imaginary roots and one real one; and (3) when $a^3/b^2>6.75$ there are three real, unequal roots.

Remarks and Solution by G. B. M. ZERB, A. M., Ph. D., Philadelphia, Pa., and H. V. SPUNAR, East Pittsburg, Pa.

The proposer has interchanged cases (2) and (3). When $a^3/b^2<6.75$, there are three real unequal roots; when $a^3/b^2>6.75$ there are two imaginary and one real root.

Let $m^3=\pm\frac{1}{2}b+\sqrt{\frac{b^2}{4}-\frac{a^3}{27}}$, $n^3=\pm\frac{1}{2}b-\sqrt{\frac{b^2}{4}-\frac{a^3}{27}}$, and ω and ω^2=the

cube roots of unity. Then the roots are $m+n$, $\omega m+\omega^2 n$, $\omega^2 m+\omega n$. When $b^2/4=a^3/27$, or $a^3/b^2=\frac{27}{4}=6.75$, $m=n=\pm\frac{1}{2}\sqrt[3]{(4b)}$.

$\therefore m+n=\pm\sqrt[3]{(4b)}$, $\omega m+\omega^2 m=\mp\frac{1}{2}\sqrt[3]{(4b)}$, $\omega^2 m+\omega n=\mp\frac{1}{2}\sqrt[3]{(4b.)}$

When $b^2/4<a^3/27$, or $a^2/b^2<\frac{27}{4},=6.75$, $\sqrt{(b^2/4-a^3/27)}$ is imaginary.

Let $m=u+\sqrt{(-1)}v$, $n=u-\sqrt{(-1)}v$.

$\therefore m+n=u+v$, $\omega m+\omega^2 n=-u-v\sqrt{3}$, $\omega^2 m+\omega n=-u+v\sqrt{3}$, all real and unequal.

When $a^3/b^2>\frac{27}{4},=6.75$, $\sqrt{\frac{b^2}{4}-\frac{a^3}{27}}$ is real.

$\therefore m+n$ is real, and $\omega m+\omega^2 n$, $\omega^2 m+\omega n$ are imaginary.

J. W. Clawson, of Collegeville, Pa., referred to Burnside and Panton's *Theory of Equations*, Vol. I, §§42, 43. Discussions of this problem are to be found in nearly all texts on the Theory of Equations.

296. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philahelphia, Pa.

Sum the series, $1+\frac{1}{6}+\frac{1}{20}+\frac{1}{50}+\frac{1}{105}+\frac{1}{196}+\frac{1}{336}+\ldots$

Solution by A. R. MAXSON, A. M., Columbia University, New York.

In the series 1, 6, 20, 50, 105, 196, 336, ..., the successive orders of differences are,

$$\begin{array}{c} 5,\ 14,\ 30,\ 55,\ 91,\ 140,\ \ldots \\ 9,\ 16,\ 25,\ 36,\ 49,\ \ldots \\ 7,\ 9,\ 11,\ 13,\ \ldots \\ 2,\ 2,\ 2,\ \ldots \\ 0,\ 0,\ \ldots \end{array}$$

The nth term is then

$$1+5(n-1)+\frac{9}{2!}(n-1)(n-2)+\frac{7}{3!}(n-1)(n-2)(n-3)$$

$$+\frac{2}{4!}(n-1)(n-2)(n-3)(n-4)=\frac{n}{12}(n+2)(n+1)^2.$$

The nth term of the given series is then $\frac{12}{n(n+2)(n+1)^2}$, which can be written $\left(\frac{6}{n}+\frac{6}{n+1}\right)-\left(\frac{6}{n+1}-\frac{6}{n+2}\right)-\frac{12}{(n+1)^2}$.

Taking now u_r as the rth term of the original series, we have

$$u_1=\left(\frac{6}{1}+\frac{6}{2}\right)-\left(\frac{6}{2}+\frac{6}{3}\right)-12.\frac{1}{2^2},$$

$$u_2=\left(\frac{6}{2}+\frac{6}{3}\right)-\left(\frac{6}{3}+\frac{6}{4}\right)-12.\frac{1}{3^2},$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$u_{n-1}=\left(\frac{6}{n-1}+\frac{6}{n}\right)-\left(\frac{6}{n}+\frac{6}{n+1}\right)-12.\frac{1}{n^2}.$$

By addition, we have $\sum\limits_{r=1}^{r=n-1} u_r=9-\dfrac{6}{n}-\dfrac{6}{n+1}+12-12\sum\dfrac{1}{n^2}$. For the sum to infinity we have $\sum\limits_{r=1}^{r=\infty} u_r=21-12.\dfrac{\pi^2}{6}=21-2\pi^2$, on remembering that $\sum\limits_1^\infty\dfrac{1}{n^2}=\dfrac{\pi^2}{6}$.

Also solved by G. B. M. Zerr, J. W. Clawson, and H. V. Spunar.

297. Proposed by W. J. GREENSTREET, Marling School, Stroud, England.

If a, b, c, d, f, g, h are all real, and a, $ab-h^2$, $abc+2fgh-af^2-bg^2-ch^2$ are all positive, show that b, c, $bc-f^2$, and $ca-g^2$ are also positive.

I. Solution by C. R. MacINNIS, Princeton, N. J.

Since both a and $ab-h^2$ are positive, b must be positive.

$$abc+2fgh-af^2-bg^2-ch^2\equiv\frac{(ab-h^2)(bc-f^2)-(hf-bg)^2}{b}.$$

Since the whole expression is positive and both b and $ab-h^2$ are also positive, $bc-f^2>0$. Hence $c>0$. Similarly,

$$abc+2fgh-af^2-bg^2-ch^2\equiv\frac{(ab-h^2)(ca-g^2)-(hg-af)^2}{a},\text{ and } ca-g^2>0.$$

II. Solution by A. F. CARPENTER, Hastings, Nebr.

Since $ab-h^2$ is positive $ab>h^2$, and since h is real, h^2 is positive. Then ab, which is greater than h^2, is positive. But a is positive; hence b is positive.

Now $b(abc+2fgh-af^2-bg^2-ch^2)=(ab-h^2)(bc-f^2)-(bg-fh)^2$; that is, $(bc-f^2)(ab-h^2)=b$ (a positive quantity) $+(bg-fh)^2=a$ positive quantity, and since $ab-h^2$ is positive, $(bc-f^2)$ is positive.

Again, $a(abc+2fgh-af^2-bg^2-ch^2)=(ab-h^2)(ca-g^2)-(af-hg)^2$, and it follows as before that $(ca-g^2)$ is positive.

From $bc-f^2$=a positive quantity, or $ca-g^2$=a positive quantity, it is readily seen that c is positive.

Also solved by H. V. Spunar and G. B. M. Zerr.

GEOMETRY.

330. Proposed by J. J. QUINN, Ph. D., New Castle, Pa.

A line pivoted at the origin revolving with a constant angular velocity, intersects another moving parallel to the Y-axis with a constant linear velocity. (1) Find the locus of their intersection when the ratio of their velocities is as $m:n$ referred to a quadrant and a radius, respectively. (2) Assume $m=3$ and $n=2$, and apply to the trisection of an angle. (3) Under what conditions will this curve become a quadratrix? (4) Name the curve.

Solution by H. V. SPUNAR, M. and E. E., East Pittsburg, Pa.

The angular velocity of the intersection point P, due to the rotation of the radius vector ρ, is constant, say $v_1=d\theta/dt$, and that linear due to the constant velocity of the moving line in the direction of X-axis is $v_2=dx/dt$.

(1) Assuming the ratio of the velocities $v_1/v_2=m/n$, the locus of the point P is

$$\frac{m}{n}=\frac{dv}{dx}, \text{ or } X=\frac{n}{m}\theta+\theta_1.$$

Letting the starting point be the origin, we have in polar coordinates, $\rho\cos\theta=\dfrac{n}{m}\theta$.

The curve may be applied without any difficulty to the multisection as well as to the trisection of an angle.

(2) To apply the curve "$\rho\cos\theta=2(\theta/3)$" to the trisection of the given angle θ. Draw OP at an angle θ (*i. e.*, the angle to be trisected) and PP_1 perpendicular to X-axis; then $OP_1=\rho\cos\theta=2(\theta/3)=x_1$; since putting $\theta=3\phi$, $\rho\cos3\phi=3[2(\theta/3)]=3x_1$. Trisecting the linear abscissae $OP_1=x_1$ and erecting at $Q_1(x_1/3)$, the perpendicular to the X-axis, cutting the curve at Q, then drawing QO, we obtain $\measuredangle QOX=\phi=(\theta/3)$.

(3) Let in $r\sin\phi=n\phi$ (Dinostrates' Quadratrix), $\phi=(\frac{1}{2}\pi-\theta)$=the complement of the angle θ. Then $\rho\cos\theta=\dfrac{n}{m}\left(\dfrac{\pi}{2}-\theta\right)$.

(4) Hence the name of the curve may be Complementary Quadratrix.

Also solved very neatly by G. B. M. Zerr.

331. Proposed by C. N. SCHMALL, New York City.

The center of two spheres radii r_1, r_2, are at the extremities of a straight line $2a$ on which as a diameter a circle is described. Find a point on the circumference from which the greatest portion of spherical surface is visible.

Solution by G. B. M. ZERR, A. M., Ph.D., Philadelphia, Pa.; A. H. HOLMES, Brunswick, Me.; and J. SCHEFFER, A. M., Hagerstown, Md.

Let $[x, \sqrt{(a^2-x^2)}]$ be the coordinates of the point referred to the mid-point of $2a$ as origin. Then $\sqrt{[2a(a-x)]}$ and $\sqrt{[2a(a+x)]}$ are the distances of the centers of the spheres from this point, where $\sqrt{[2a(a-x)]}$ is the distance to the center of sphere radius $r_1<r_2$.

$$\therefore 2\pi r_1^2\left(1-\frac{r_1}{\sqrt{[2a(a-x)]}}\right) \text{ and } 2\pi r_2^2\left(1-\frac{r_2}{\sqrt{[2a(a+x)]}}\right)$$

are the portions of the respective spherical surfaces visible from the point.

$$\therefore 2\pi\left(r_1^2-\frac{r_1^3}{\sqrt{[2a(a-x)]}}+r_2^2-\frac{r_2^3}{\sqrt{[2a(a+x)]}}\right)=\text{a maximum.}$$

$$\therefore \frac{r_2^3}{\sqrt{(a+x)}}=r_1^3/\sqrt{(a-x)}. \quad \therefore x=\frac{a(r_2^6-r_1^6)}{r_2^6+r_1^6}, \; \sqrt{(a^2-x^2)}=\frac{2ar_1^3r_2^3}{r_1^6+r_2^6}.$$

$$\text{Visible surface}=2\pi\left(r_1^2+r_2^2-\frac{\sqrt{(r_2^6+r_1^6)}}{a}\right).$$

Also solved by J. E. Sanders, C. N. Schmall, and H. V. Spunar.

CALCULUS.

256. Proposed by PROFESSOR R. D. CARMICHAEL, Anniston, Ala.

Solve the differential equation, $(1+y+2axy)dx+x(1+x)dy=0$.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

$$\frac{dy}{dx}+\frac{y(1+2ax)}{x(1+x)}+\frac{1}{x(1+x)}=0.$$

$$\therefore ye^{\int Pdx}=C+\int Qe^{\int Pdx}dx.$$

$$\int Pdx=\int\frac{1+2ax}{x(1+x)}dx=\log[x(1+x)^{2a-1}. \quad Q=-\frac{1}{x(1+x)}.$$

$$\therefore xy(1+x)^{2a-1}=C-\int(1+x)^{2a-2}dx.$$

$$\therefore \left(xy+\frac{1}{2a-1}\right)(1+x)^{2a-1}=C.$$

Also solved by G. W. Hartwell, J. Scheffer, H. V. Spunar, Francis Rust, and T. I. Wodo.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

150. Proposed by H. S. VANDIVER, Bala, Pa.

Show that for all positive integral values of n except unity, $(2n)!$ is less than $[n(n+1)]^n$. Direct proof preferred. [Unsolved problem in *Educational Times.*]

I. Solution by JACOB WESTLUND, Purdue University, Lafayette, Ind., and the PROPOSER.

If $0<k<n$, we have $(n-k)(n+k+1)<n(n+1)$, since $n^2+n-(k^2+k)<n^2+n$.

Hence, letting k run through the values 0, 1, 2, ..., $n-1$, we get $(2n)!<[n(n+1)]^n$.

II. Solution by W. F. KING, Ottawa, Canada.

$$(2n)!=(1.2.3...n)[(n+1)(n+2)...(2n)]$$

$$=[n(n-1)(n-2)...(n-\overline{n-1})]$$

$$\times[\overline{n+1}(\overline{n+1}+1)(\overline{n+1}+2)...(\overline{n+1}+\overline{n-1})].$$

$$\therefore \frac{(2n)!}{[n(n+1)]^n}=\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)...\left(1-\frac{n-1}{n}\right)\right]$$

$$\times\left[\left(1+\frac{1}{n+1}\right)\left(1+\frac{2}{n+1}\right)...\left(1+\frac{n-1}{n+1}\right)\right].$$

Taking any factor $1-\dfrac{r}{n}$ in the first bracket with the corresponding factor $1+\dfrac{r}{n+1}$ in the second bracket, their product is $\left(1-\dfrac{r}{n}\right)\left(1+\dfrac{r}{n+1}\right)$ $=1-\dfrac{r}{n(n+1)}-\dfrac{r^2}{n(n+1)}$, which is <1.

The product of each pair of terms being <1, the whole product is <1.

$$\therefore \frac{(2n)!}{[n\ (n+1)]^n}<1, \text{ and } (2n)!<[n(n+1)]^n. \quad \text{Q. E. D.}$$

III. Solution by F. H. SAFFORD, Ph. D., University of Pennsylvania.

Stating the problem in the form $\dfrac{[n(n+1)]^n}{(2n)!}>1$, it is evidently true for $n=2$, *i. e.*, $\frac{3}{2}>1$. The factor which will change the first member of the first inequality into the form in which n becomes $n+1$ is

$$F=\frac{(n+2)^{n+1}}{2\ (2n+1)n^n}=\left(\frac{n+2}{n}\right)^n\cdot\frac{n+2}{2\ (2n+1)}=\left[\left(1+\frac{2}{n}\right)^{\frac{1}{2}n}\right]^2\cdot\frac{(1+2/n)}{(4+2/n)}.$$

Since $\left(1+\dfrac{2}{n}\right)^{\frac{1}{2}n}>2$, and $\dfrac{1+2/n}{4+2/n}>\frac{1}{4}$, it follows that $F>1$, hence for successive values of n the original inequality becomes stronger.

It is of interest to notice that the limiting value of F, for $n=\infty$, is $\varepsilon^2/4$, but this is not essential to the proof.

Excellent solutions of this problem were received from G. B. M. Zerr, Frank L. Griffin, C. E. White, and O. L. Callicot.

151. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

In the recurring series, $n=0, 1, 2, 3, 4, 5, 6, 7, \ldots$
$u_n=3, 0, 2, 3, 2, 5, 5, 7, \ldots$
where the scale of relation is $u_{n+3}=u_{n+1}+u_n$, prove that u_p is always divisible by p when p is prime. Is the converse true?

Solution by the PROPOSER.

The general term of the series is $u_n=\alpha^n+\beta^n+\gamma^n$ where α, β, γ are roots of the equation $x^3-x-1=0$.

When p is prime, $(\alpha+\beta+\gamma)^p\equiv\alpha^p+\beta^p+\gamma^p$ (mod. p), and since $\alpha+\beta+\gamma=0$, we have $u_p\equiv0$ (mod. p).

I believe the converse is not true, although I have not found an example which disproves it.

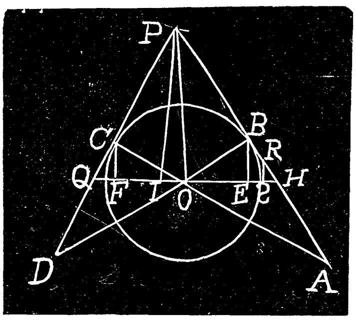
MECHANICS.

213. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

Two unequal, uniform, smoothly hinged rods are placed over a smooth vertical circle. Apply the principle of virtual work to find the condition of equilibrium in terms of the length of each rod, the diameter of the circle and the angle of either rod with the vertical.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let O be the center of the vertical circle; PD, PA, the rods jointed at P; C, B, the points of tangency of PD, PA with the circle; Q, H the intersection of the horizontal diameter with PD, PA. Let r=radius of circle, $PD=l$, $PA=m$. Draw PI, CF, BE perpendicular to HQ. Let $\angle BOE=\angle HBE=\phi$, $\angle COF=\angle FCQ=\angle IPQ=\theta$. Then $PC=PB=r\cot\frac{1}{2}(\phi+\theta)$, $CQ=r\tan\theta$, $BH=r\tan\phi$, $OI=\dfrac{r\sin\frac{1}{2}(\phi-\theta)}{\sin\frac{1}{2}(\phi+\theta)}$.



Let z=the height of the center of gravity of the system above O. Also for equilibrium the resultant of the weights of the rods must pass through O.

The weights of the rods are proportional to their lengths. $\therefore W:W'=l:m$.

$$z=\frac{l(PQ\cos\theta-\frac{1}{2}l\cos\theta)+m(PH\cos\phi-\frac{1}{2}m\cos\phi)}{l+m}$$

$$=[lr\sin\theta+lr\cos\theta\cot\tfrac{1}{2}(\phi+\theta)-\tfrac{1}{2}l^2\cos\theta+mr\sin\phi$$
$$+mr\cos\phi\cot\tfrac{1}{2}(\phi+\theta)-\tfrac{1}{2}m^2\cos\phi]/(m+l)\ldots(1).$$

Also, $(\frac{1}{2}l\sin\theta+OI)l=(\frac{1}{2}m\sin\phi-OI)m$, or
$$2r(l+m)\sin\tfrac{1}{2}(\phi-\theta)=(m^2\sin\phi-l^2\sin\theta)\sin\tfrac{1}{2}(\phi+\theta)\ldots(2).$$

Differentiating (1) and (2), we get

$$[l^2\sin\theta\sin^2\tfrac{1}{2}(\phi+\theta)-r(l+m)\cos\phi]d\theta$$
$$+[m^2\sin\phi\sin^2\tfrac{1}{2}(\phi+\theta)-r(l+m)\cos\theta]d\phi=0\ldots(3).$$

$$\frac{2l^2\cos\theta\tan\frac{1}{2}(\phi+\theta)+l^2\sin\theta-m^2\sin\phi-2r(l+m)\cos\frac{1}{2}(\phi-\theta)}{\cos\frac{1}{2}(\phi+\theta)}d\theta$$

$$=\frac{2m^2\cos\phi\tan\frac{1}{2}(\phi+\theta)-l^2\sin\theta+m^2\sin\phi-2r(l+m)\cos\frac{1}{2}(\phi-\theta)}{\cos\frac{1}{2}(\phi+\theta)}d\phi\ldots(4).$$

Eliminating $d\theta$, $d\phi$ between (3) and (4) we get, after reducing,

$$(m^2\sin\phi-l^2\sin\theta)^2\sin^2\tfrac{1}{2}(\phi+\theta)-4l^2m^2\sin^4\tfrac{1}{2}(\phi+\theta)$$
$$+r(l+m)(l^2\sin\theta-m^2\sin\phi)(\cos\theta-\cos\phi)+[2r(l+m)(l^2\cos^2\theta$$
$$+m^2\cos^2\phi)+r(l+m)(m^2\sin\phi+l^2\sin\theta)(\sin\phi+\sin\theta)]$$
$$\times\tan\tfrac{1}{2}(\phi+\theta)-4r^2(l+m)^2\cos^2\tfrac{1}{2}(\phi-\theta)=0\ldots(5).$$

(2) and (5) determine the equilibrium. If $l=m$, $\phi=\theta$, and we get

$$l^2\sin^4\theta\cos\theta-2rl\sin\theta+4r^2\cos\theta=0.$$
$$(l\sin^3\theta-2r\cos\theta)(l\sin\theta\cos\theta-2r)=0.$$

$l\sin^3\theta=2r\cos\theta$ is the condition for equilibrium when the rods are equal.

214. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

An inelastic particle is projected in a direction BD from B in a straight line AB. It strikes a rigid line AD in D and returns to AB at C. Find AC/AB, and show on *a priori* ground that this ratio is independent of the velocity of projection.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let e be the elasticity of the rigid line AD, DR a prolongation of AD. Let $AB=a$, $\angle DAB=\beta$, $\angle DBC=\delta$, $\angle RDC=\theta$. Then $\angle ADB=\delta-\beta$, $\tan\theta=e\tan(\delta-\beta)$. $\therefore\theta=\tan^{-1}[e\tan(\delta-\beta)]$, and is known.

Since θ is independent of the velocity of projection, AC/AB is independent of this velocity.

$$AD:a=\sin\delta:\sin(\delta-\beta).\quad\therefore AD=\frac{a\sin\delta}{\sin(\delta-\beta)}.$$

$$AC:AD=\sin\theta:\sin(\theta-\beta).\quad\therefore AC=\frac{AD\sin\theta}{\sin(\theta-\beta)}=\frac{a\sin\delta\sin\theta}{\sin(\delta-\beta)\sin(\theta-\beta)}.$$

$$\therefore\frac{AC}{a}=\frac{AC}{AB}=\frac{\sin\delta\sin\theta}{\sin(\delta-\beta)\sin(\theta-\beta)}.$$

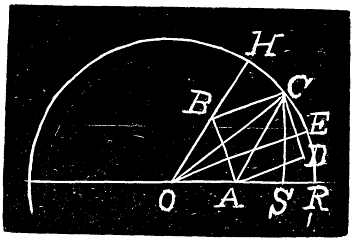
AVERAGE AND PROBABILITY.

193. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

What is the average area of all squares that may be inscribed in a given sector of a circle, a diagonal of the square being parallel to a random line across the sector?

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

The following solution is for a sector less than a quadrant. Larger sectors would require a separate solution.



Let ROH be the given sector, $ABCD$ the inscribed square, C in the arc, A and B each in a radius; A in OR, B in OH; AC the diagonal representing the random direction of the line; CS, perpendicular from C on OR. All directions will be included between AC in OR and AC parallel to the bisector of sector.

Let $AB=x$, $OC=OR=r$, $\angle HOR=\beta$, $\angle CAR=\theta$. Then $AC=x\sqrt{2}$, $AS=x\sqrt{(2)}\cos\theta$, $CS=x\sqrt{(2)}\sin\theta$, $OA=x\sin(\frac{1}{4}\pi+\theta-\beta)/\sin\beta$.

$\therefore(x\sin(\frac{1}{4}\pi+\theta-\beta)/(\sin\beta+x\sqrt{(2)}\cos\theta)^2+2x^2\sin^2\theta=r^2$.

$$\therefore x^2=\frac{r^2\sin^2\beta}{\sin^2(\frac{1}{4}\pi+\theta-\beta)+2\sqrt{(2)}\sin\beta\cos\theta\sin(\frac{1}{4}\pi+\theta-\beta)+2\sin^2\beta}$$

$$=\frac{2r^2\sin^2\beta}{2+\sin2\beta-\cos2\beta+(1+\sin2\beta)\sin2\theta+(\cos2\beta-1)\cos2\theta}.$$

The limits of θ are $\frac{1}{4}\pi$ and $\frac{1}{4}\pi+\frac{1}{2}\beta=\theta_1$.

$$\therefore \triangle=\frac{2}{\beta}\int_{\frac{1}{4}\pi}^{\theta_1} x^2 d\theta = \text{average area.}$$

$$\therefore \triangle=\frac{4r^2\sin^2\beta}{\beta(c-1)}\left[\tan^{-1}\left(\frac{c-\sqrt{(2c-1)}}{c-1}\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\beta-\tfrac{1}{2}\alpha)\right)\right.$$
$$\left.-\tan^{-1}\left(\frac{c-\sqrt{(2c-1)}}{c-1}\tan(\tfrac{1}{4}\pi-\tfrac{1}{2}\alpha)\right)\right],$$

where $c=2+\sin 2\beta-\cos 2\beta$, $1+\sin 2\beta=\sqrt{(2c-1)}\sin\alpha$,
$\cos 2\beta-1=\sqrt{(2c-1)}\cos\alpha$.

If $\beta=\frac{1}{4}\pi$, $\triangle=\frac{4r^2}{\pi}\{\tan^{-1}\tfrac{1}{2}[\sqrt{10}-3][3-\sqrt{5}]+\tan^{-1}\tfrac{1}{2}[3-\sqrt{5}][\sqrt{5}-2]\}$.

$$\therefore \triangle=.1933r^2.$$

If we accept the usual definition of inscription of geometric figures, viz., the vertices of the inscribed figure shall lie on the boundary of the other figure, the above problem is impossible; for a square cannot be inscribed in a sector so that its diagonal shall be parallel to a random direction. For a sector less than a quadrant three squares can be inscribed, and only three. The average area would be one-third of the sum of the areas of these three squares. The above solution satisfies the case when three vertices of the square lie on the boundary of the sector. The author of the problem may have wished the solution to cover the case when the vertices lie wholly within the sector, the limiting case being when three vertices lie on the boundary. In this case he should not have used the term "inscribed." ED. F.

MISCELLANEOUS.

172. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

If ϕ and ψ are small angles, show that an approximate value of ϕ/ψ is

$$\tfrac{2}{3}\frac{\sin\phi}{\sin\psi}+\tfrac{1}{3}\frac{\tan\phi}{\tan\psi}-\tfrac{1}{180}(\phi^2-\psi^2)(9\phi^2-\psi^2).$$

II. Solution by the PROPOSER.

Neither Prof. Zerr nor Mr. Greenwood seems to have carried the expansion quite far enough. When this is done the approximation stated will be found to be perfectly correct. The following relations will show this to be the case.

$$\tfrac{2}{3}\frac{\sin\phi}{\sin\psi}=\tfrac{2}{3}\left[\phi-\frac{\phi^3}{6}+\frac{\phi^5}{120}\right]\left[\psi-\frac{\psi^3}{6}+\frac{\psi^5}{120}\right]^{-1}, \text{ approximately,}$$

$$=\tfrac{2}{3}\frac{\phi}{\psi}\left[1-\left(\frac{\phi^2}{6}-\frac{\phi^4}{120}\right)\right]\left[1-\left(\frac{\psi^2}{6}-\frac{\psi^4}{120}\right)\right]^{-1}\cdots$$

$$=\tfrac{2}{3}\frac{\phi}{\psi}\left[1+\frac{\psi^2}{6}-\frac{\phi^2}{6}+\frac{\psi^4}{36}-\frac{\psi^4}{120}+\frac{\phi^4}{120}-\frac{\phi^2\psi^2}{36}\right]\ldots$$

So, $\tfrac{1}{3}\dfrac{\tan\phi}{\tan\psi}=\tfrac{1}{3}\dfrac{\phi}{\psi}\left[1+\left(\dfrac{\phi^2}{3}+\dfrac{2\phi^4}{15}\right)\right]\left[1+\left(\dfrac{\psi^2}{3}+\dfrac{2\psi^4}{15}\right)\right]^{-1}\ldots$

$$=\tfrac{1}{3}\frac{\phi}{\psi}\left[1+\frac{\phi^2}{3}-\frac{\psi^2}{3}-\frac{2\psi^2}{15}+\frac{\psi^4}{9}+\frac{2\phi^4}{15}-\frac{\phi^2\psi^2}{9}\right]\ldots$$

Their sum$=\dfrac{\phi}{\psi}\left[1+\dfrac{\psi^4}{180}-\dfrac{10\phi^2\psi^2}{180}+\dfrac{9\phi^4}{180}\right]\ldots$

$$=\frac{\phi}{\psi}+\frac{\phi}{\psi}\cdot\tfrac{1}{180}[\phi^2-\psi^2][9\phi^2-\psi^2].$$

In Mr. Greenwood's solution it will be noticed that $\dfrac{\tan\phi}{\tan\psi}$ is *not AB* as stated.

The relation $\tan\phi=\phi+\frac{1}{3}\phi^3+\frac{2}{15}\phi^5+\frac{17}{315}\phi^7\ldots$ is readily obtained from

$$\tan\phi=\left[\phi-\frac{\phi^3}{6}+\frac{\phi^5}{120}-\frac{\phi^7}{5040}+\ldots\right]\left[1-\frac{\phi^2}{2}+\frac{\phi^4}{24}-\frac{\phi^6}{720}+\ldots\right]^{-1}.$$

The second factor$=\left[1+\left(\dfrac{\phi^2}{2}-\dfrac{\phi^4}{24}+\dfrac{\phi^6}{720}\right)+\left(\dfrac{\phi^2}{2}-\dfrac{\phi^4}{24}\right)^2+\left(\dfrac{\phi^2}{2}\right)^3+\ldots\right]$

and on multiplying out the expansion is obtained to any power we require.

PROBLEMS FOR SOLUTION.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

157. **Proposed by A. H. HOLMES, Brunswick, Maine.**

Find integral values for m and n in $64m^2n^2(m^2-n^2)^2+(m^2+n^2)^4=\square$.

158. **Proposed by J. EDWARD SANDERS, Reinersville, Ohio.**

Find positive rational values of a and b in the equation $x^4-2ax^2+x+a^2-b=0$, that will make each of the roots (all different) rational numbers.

MISCELLANEOUS.

179. Proposed by FRANCIS RUST, C. E., Allegheny, Pa.

Let S represent the area swept over by the radius vector, counted from the perihelion; m represent the *mean anomaly*, and e represent the eccentricity of the orbit. Prove: $2S=m\sqrt{(1-e^2)}$.

NOTES AND NEWS.

Doctor J. W. Young, formerly preceptor at Princeton University, is now assistant professor of Mathematics at the University of Illinois.

Doctor R. L. Moore, formerly instructor at Princeton University, is now instructor in Mathematics at Northwestern University, Evanston, Ill.

Doctor Max Mason, formerly assistant professor at Yale University, is now associate professor of Mathematics at the University of Wisconsin.

Doctor N. J. Lennes, formerly instructor at the Massachusetts Institute of Technology, is now instructor in Mathematics at Brown University.

Doctor Gilbert A. Bliss, formerly assistant professor at Princeton University, has been appointed associate professor of Mathematics at Chicago University.

Mr. E. J. Moulton, formerly a graduate student at the University of Chicago, is now in charge of the Department of Astronomy at Pritchett College, Glasgow, Mo.

The Summer Meeting of the American Mathematical Society was held at the University of Illinois, September 10th and 11th, 1908. The attendance was good and the meeting highly successful.

Doctor T. E. McKinney, formerly in charge of the Department of Mathematics at Wesleyan University at Middleton, Connecticut, is now professor of Mathematics in the State University of South Dakota at Vermillion, S. D.

Mr. F. H. Hodge, formerly instructor in the Undergraduate Department of Clark University and more recently a graduate student at the University of Chicago, is now professor of Mathematics at Parsons College, Fairfield, Iowa.

The annual conference of the University of Chicago with cooperating high schools of the Middle West takes place in Chicago, November 13th and 14th, 1908. In connection with the general program, departmental confer-

ences are held and among these is a conference in Mathematics which always proves of great interest to those who are teaching in this line. The subject this year is "Improvement in the Teaching of Geometry." The chief speaker is Principal H. W. Hurt of Oskaloosa, Iowa.

The annual meeting of the Kansas Association of Science and Mathematics Teachers will be held at Lawrence, Kansas, on November 27th and 28th. An address will be given before this association on the "Preparation of the Teacher of Mathematics," by Professor H. E. Slaught, of the University of Chicago. The Southwestern Section of the American Mathematical Society will meet in Lawrence at the same time, but the sessions will be arranged so as not to conflict with those of the State society.

The annual meeting of the Central Association of Science and Mathematics Teachers will be held at the Englewood High School, Chicago, Ill., November 27th and 28th, 1908. The program of the Mathematics section promises to be of special interest. It will include a revised report of the Committee on Algebra of last year together with a report of a new committee on "Correlated Mathematics for Secondary Schools." This association ought to include every secondary teacher of Mathematics in the Middle West.

The annual meeting of the University of Illinois with the high schools of the State will take place at Urbana on November 20th and 21st, 1908. Of especial interest to teachers of Mathematics is the report of the commission which was appointed last year to draft a syllabus of algebra for the high schools. The committee consists of Professor H. L. Rietz, University of Illinois, chairman; Professor H. E. Slaught, University of Chicago, Mr. Manners, of East St. Louis, Illinois, Louis Omer, Oak Park, Illinois, and Jacob Meyer, formerly of Dixon, Illinois,

BOOKS.

General Physics. An Elementary Text-Book for Colleges. By Henry Crew, Ph. D., Fayerweather Professor of Physics in Northwestern University. 8vo. Cloth sides and leather back, xi+522 pages. Price, $2.75. New York: The Macmillan Co.

The author in his preface states that his purpose in writing this book was three-fold, viz., (1) To prepare a text-book adapted to the needs of first year students in Physics at Northwestern University, (2) To keep the treatment elementary, and yet include all the fundamental principles of Physics and, (3) To set before the student a large and compact body of truth obtained by a method which shall remain for him, throughout life, a pattern and norm of clear and correct thinking.

The book is well written, the subject matter is well arranged, and important facts

stated in clearest language. The illustrations are good and the mechanical make-up of the book is all that could be desired.

The book will be found very valuable for all college teachers of Physics who wish a good book for first year Physics students. B. F. F.

A Short University Course in Electricity, Sound, and Light. By Robert Andrews Millikan, Ph. D., Associate Professor of Physics in the University of Chicago, and John Mills, A. M., Instructor in Physics in the Western Reserve University. 8vo. Cloth, v+388 pages. Price, $2.00. Boston and Chicago: Ginn & Co.

This book is intended to be something more than a laboratory manual. It presents a logical development, from the standpoint of theory as well as experiment, of the subjects of Electricity, Sound, and Light. The course here outlined and treated is analytical rather than descriptive, although no mathematics beyond Trigonometry is presupposed. The book will be found very valuable to those teachers who prefer text-books treating on different subjects in Physics rather than a single text. It is well gotten up and the publishers have added to the attractiveness of the work by presenting it to the public in good clear type and substantial binding. B. F. F.

Practical Elementary Algebra. By Joseph V. Collins, Ph. D., Professor of Mathematics, State Normal School, Stevens Point, Wisconsin. 8vo. Cloth sides and leather back, 420 pages. Price, $1.00. New York and Chicago: American Book Co.

The author of this work being touched by the wave of interest in the teaching of Mathematics which has swept over the United States during the past four or five years, has attempted to write a book which will meet the demands made by the various organizations of teachers, which organizations propogated the wave which is now carrying the teaching of Mathematics on its crest. The literature of the various organizations of Mathematics teachers has been carefully studied by the author and he has endeavored to utilize as far as possible all practical suggestions. He has searched the whole field of exercises for practical problems and has introduced a considerable number of new ones. This book may be classed among the best of those written during the last few years. B. F. F.

An Algebra for Secondary Schools. By E. R. Hedrick, Professor of Mathematics, The University of Missouri. 8vo. Cloth and leather back, x+421 pages. Price, $1.00. New York and Chicago: American Book Co.

This book meets the entrance requirements of American colleges and universities, though it was written with a view to meet the demands of those students for whom the high school course is the last. The author has embodied the best views, both radical and conservative, expressed in recent reports of committees on the teaching of Algebra. The language of the book is simple and conversational. The problems are numerous and represent a wide range of subjects. Frequent use is made of the graph. B. F. F.

Differential and Integral Calculus. By Daniel A. Murray, Ph. D., Professor of Applied Mathematics in McGill University. 8vo. Red cloth, xviii+491 pages. Price, $2.00. New York: Longmans, Green, & Co.

This book is mainly made up of the material from Dr. Murray's *Infinitesimal Calculus*, though much additional matter especially beneficial to engineering students has been incorporated. The notion of anti-differentiation is presented and exercises in this operation appear early. Later the operation of integration is taken up and treated as a process of summation. The book is well written and is worthy of wide adoption as a text. B. F. F.

Paul Jean Joseph Barbarin

THE AMERICAN MATHEMATICAL MONTHLY.

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BIOGRAPHICAL SKETCH OF PAUL BARBARIN.

By DR. GEORGE BRUCE HALSTED, Greeley, Colorado.

Paul Jean Joseph Barbarin, one of the greatest of living geometers, was born October 20, 1855, in Tarbes (High Pyrinees). He had the almost indispensable advantage for mathematical achievement of a very early start, which came about as follows: Though his father was a professor of mathematics, and tried early to awaken in him a taste for the sciences, yet our Professor Barbarin slighted these lessons until after he had taken, when scarcely 16 years old, his degree of Bachelor of Letters. The following year, to please his father, he consented to take a course in elementary mathematics, when the love of science quickly developed, and he expressed a wish to attend the Polytechnic School and the Normal School. He entered the Polytechnic for a short time, but changed to the more congenial Normal, where at $19\frac{1}{2}$ years of age he settled down to prepare for his life work. He studied under Briot, Bouquet, Tannery, and Darboux.

His course finished, he became professor of mathematics at Niza, then at the School of St.-Cyr of the Lyceüm of Toulon. Since 1891 he has been professor at the Lyceum of Bordeaux. He married a highly intellectual lady, born at Reichshofen in Alsace, and her mastery of German, English, Spanish, and Portugese has given essential help in his writings. Our Professor is noted for his devotion to the river and sea and all aquatic sports, but above all to music. Both he and his wife play brilliantly upon the violoncello, and their charming home is a center for the musical cult.

Notwithstanding the heavy draft of his teaching upon his energies, he has been extraordinarily productive and original as a geometer, his discoveries in non-Euclidean geometry being particularly brilliant. The report on his works by Professor Mansion on the occasion of the third award of the Lobachevski prize, where he was second only to Hilbert, I have given in full in English in *Science*, Vol. XX, pp. 353-367. From this a single sentence may be here reproduced: "Non-Euclidean geometry owes to M. Barbarin (1) fundamental properties of the plane trirectangular quadrilateral; (2) the discovery of Riemannean equidistant straights; (3) the complete

classification of non-Euclidean conics and quadrics; (4) the most intuitive formula that we know for the determination of volumes, with a remarkable application to the tetrahedron; (5) finally and above all, the beautiful general theorem cited above on the geodesics of tubes and pseudospheres, in the three geometries."

Professor Barbarin is like Poincaré in adding to his creative power the gift of brilliant exposition. An example is his beautiful little book, "La Géométrie non-Euclidienne." The first edition of this I reviewed in *Science*, Vol. XV, pp. 984-988. A second edition has now been issued by Gauthier-Villars, 8vo, 91 pages, greatly improved; for example, by introducing the single elliptic geometry so strangely unmentioned in the first edition. This delightful little treatise is a perfect gem.

Of late Professor Barbarin, already noted as worthy successor of Hoüel, who made Bordeaux sacred ground for non-Euclidean geometry, has exhibited, like his beloved predecessor, a genius for translating. His translation, La Spherique non-euclidienne, in L' Enseignement mathématique, No. 2, 10^{e} année, Mars 1908, pp. 97-111, is a marvel of elegance, clearness, and accuracy.

The celebrated Société des Sciences de Bordeaux honored itself in 1905 when it elected Professor Barbarin as its president.

A List of the Principal Memoirs and Works of P. Barbarin.

1. Note sur les coordonnèes bipolaires. (Nouvelles Annales). 1882.
2. Note sur la droite de Simpson. (Mathesis). 1882.
3. Sur l' Herpolhodie: N. A. 1885.
4. Sur un Systeme d' Equations. Revue de Speciales. 1894.
5. Normales généralisées. Revue de Speciales. 1894.
6. Systemes isogonaux du triangle. A. F. A. S. 1896.
7. Triangles dont les bissectrices ont des longueurs données. M. 1896.
8. Constructions Spheriques. M. 1899.
9. Etudes de géométrie Analytique non-Euclidienne. Bruxelles. 1900.
10. Géométrie Infinitesimal non-Euclidienne. Lisbonne. 1901.
11. V^{e} Livre de Metagéométrie. M. 1901.
12. Polygones réguliers Spheriques. Le Matematiche. 1902.
13. Cosegments et Volumes. (Memoires de Bordeaux). 1902.
14. La Geometrie non-Euclidienne. (Scientia). Paris. 1902.
15. Calculs abrégés de Sinus et Cosinus. (Memoires de Bordeaux). 1904.
16. Bilatères et Trilatères. M. 1902.
17. Considerations sur la forme de l' Espace. (Enseignement Math.). 1902.
16. Spherique non-Euclidienne de G. B. Halsted. (Enseignement Mathématique). 1908.
19. Recueil de Calculs Logarithmiques. Paris, Nony. 1893.
20. Complements sur les Courbes usuelles. Paris, Nony. 1898.

DEFINITIONS OF THE TERM "MATHEMATICS."

By DR. G. A. MILLER, University of Illinois.

As science advances many laws are found to be less simple than they were at first supposed to be and many terms acquire a breadth of meaning far beyond that assigned to them when they were first employed. In an age of such great scientific activity as the present, it might appear especially futile to try to define terms covering great fields of scientific inquiry, in view of the fact that each important new development tends to throw new light on the term under which it is classed. Moreover, as these developments proceed, the sciences tend more and more to grow into each other, and this complicates very seriously the formulating of satisfactory definitions. Broad definitions are not apt to meet with much opposition unless they encroach on the territory claimed by other terms by right of first exploration. As illustrative of this thought we need only point to the vague notions as regards the boundaries between arithmetic, algebra, and geometry.

Notwithstanding these serious difficulties, definitions of general terms frequently furnish most inspiring view-points of broad domains. Such a term as "Mathematics," for instance, is continually becoming more replete with meaning for the growing mathematician, and, if he meets a definition which is just suited to his advancement, it tends not only to give him new pleasure but also to emphasize elements which had not been distinguished with sufficient clearness. Even if such a definition should not involve any element of novelty, it may give us pleasure by supporting a view which has been entertained by us for a long time. Probably few young mathematicians have read or re-read the remarks by Professor Bôcher on "Old and New Definitions of Mathematics,"* without arriving at a clearer conception of some broad principle relating to mathematical thought.

The main object of the present note is a brief consideration of the statement: *Mathematics is the science of saving thought:*† In the first place, we may compare the scope of this statement with the well known definition which Benjamin Peirce gave in his Linear Associative Algebra (1870), viz.: *Mathematics is the science which draws necessary conclusions.* If, by saving thought, we mean the most economical use of thought these two definitions appear to have practically the same scope. For if we can draw a necessary conclusion in regard to a subject this conclusion may be drawn once for all by the same individual, and thereafter when circumstances arise which fulfil the conditions leading to the first conclusion the same result may be assumed without going through the various steps of reasoning. On the other hand, if we arrive at a definite conclusion without expending thought on all

*Bulletin of the American Mathematical Society, Vol. 11 (1904), p. 115.

†Popular Science Monthly, Vol. 73 (1908), p. 321.

its details, it is essential that some of the intermediate steps were omitted in view of the fact that they could be reached by drawing necessary conclusions.

Having observed that the definitions of the preceding paragraph have practically the same scope, the question arises whether this scope is not too comprehensive. It is easily seen that it includes developments which are not generally classed with Mathematics. It has been claimed that such developments are the result of the tendency of sciences to grow into each other and to encroach on domains which are not strictly their own. As is well known, some of the sciences which started out as non-mathematical, in their later developments availed themselves of mathematical results with great profit. A very striking instance is furnished by the "Phase Rule" of Professor Gibbs, which now occupies such a fundamental position in Physical Chemistry. Fairness, however, demands that the mathematician should not lay claim to territory first explored by other sciences even if it joins domains universally acceded to him and could have been properly claimed if the mathematician had been first to occupy it, or if the exploration had been accomplished by mathematical means.

In view of these facts it might be best to regard the so-called definitions under consideration as incomplete definitions, expressing concepts which dominate mathematical thought more completely than that of the other sciences. In his address before the recent International Mathematical Congress held at Rome, Poincaré developed the thought that *the role of science is to produce economy of thought,* which he attributed to Mach. He said that the savage calculates with his fingers or with little pebbles. In teaching children the multiplication table we save them innumerable manipulations with pebbles. In olden times someone recognized, with pebbles or otherwise, that 6 times 7 is 42 and he had the idea to note the result, and on this account we do not have to begin over again. This one did not lose his time even if he calculated only for pleasure; his operation took him only two minutes, while it would have required two billion minutes if a billion men would have been compelled to begin where he began. The importance of a fact is therefore measured by the quantity of thought which it enables us to economize.*

We desire to consider briefly another expression, viz., *Mathematics is a tool.* There is evidently some truth in this. A tool enables us to use physical energy more effectively just as Mathematics enables us to use thought with greater efficiency. On the other hand a tool is a simple instrument while Mathematics is very complex. It would be better to say that Mathematics consists of ten thousand tools, but this would not convey a complete idea, for Mathematics does not only develop a large number of simple tools but it especially emphasizes the putting together of these tools into powerful thought machines. The training in the construction of such

Bulletin des Sciences Mathematiques, Vol. 32 (1908), p. 171.

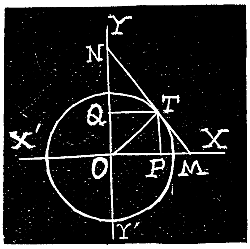
machines is one of the greatest importance to those who expect to employ Mathematics intelligently in Physics or Engineering. If we are not mistaken Professor Slichter had practically the same thought in mind when he said: "It grates on me to hear Mathematics spoken of as a tool. Mathematics is to the engineer a basal science and not a tool. The spirit of that science is of more value to the engineer than the particular things that can be accomplished. The engineer need not be a mathematician, but he needs to think mathematically, and, to my mind, he needs the power of mathematical thought more than skill in manipulating a few mathematical tools in a mechanical fashion."*

*Science, Vol. 28 (1908), p. 263.

ON THE REPRESENTATION OF THE TRIGONOMETRIC FUNCTIONS BY LINES.

By R. D. CARMICHAEL, Alabama Presbyterian College.

The representation of the trigonometric functions dealt with in this note is in some respects different from that usually employed; it has certain advantages which appear to make it superior to other methods of effecting this representation.



Let the circle whose center is O (see figure) have the radius unity; and let O be the origin of the rectangular axes XX' and YY'. Let the angle $MOT=x$ be formed by the radius revolving counter-clockwise from OX to any position OT. Draw TP and TQ perpendicular, respectively, to OX and OY. At T draw a tangent to the circle cutting the X-axis in M and the Y-axis in N. Then in whatever quadrant OT may lie we have

$$\sin x = OQ, \quad \sec x = OM, \quad \tan x = TM,$$
$$\cos x = OP, \quad \csc x = ON, \quad \cot x = TN.$$

The tangent and the cotangent are measured from the point of tangency; the other functions from the center of the circle. Thus, all functions are measured from an extremity of the revolving radius.

It should be noticed that in any quadrant the tangent is that portion of the tangent line intercepted between the point of tangency and the X-axis, while the cotangent is intercepted between the point of tangency and the Y-axis. *This is the conception of Analytics.* The secant and the

cosine are measured along the X-axis, while the sine and the cosecant are measured along the Y-axis. The following facts are evident from an inspection of the figure for an angle in each quadrant:

The algebraic signs of the sine, cosine, secant, and cosecant are determined by the direction in which each is measured from O in accordance with the usual convention of Analytics. The algebraic sign of the tangent is plus when the tangent is measured to the right of OT (as one looks from O); minus, when measured to the left. The tangent and cotangent have always the same sign.

Any two functions of an angle measured along the same line have unity for their product.

It seems to me that this representation of the functions will make it easier for the student to fix the algebraic sign of any function in any quadrant, and also to remember the group of products each equal to unity.

The method also lends itself very readily to approximate measurements of the functions for rough work. For this purpose the pupil will require a circular protractor and a "square" graduated to tenths of the unit on the inner edges of the angle. This square should have for unit the radius of the protractor. Tangents and cotangents may be read (accurately to tenths, estimated to hundredths) by laying one inner edge of the square along the radius OT which cuts off on the protractor the required angle, and then reading TM or TN according as tangent or cotangent is desired. The sine and the cosine may be read by putting the vertex of the square as at P and reading PO and PT. (It is in measuring tangents and cotangents, of course, that this method has the advantage over the ordinary methods.)

NOTE ON AN APPROXIMATION IN TRIGONOMETRY.

By G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

The question as to whether it is possible to find the angles of a triangle in terms of the sides approximately correct, without the use of tables, is one which is often asked by persons who do not know how to use tables and yet find it necessary to use the approximate values of the angles. They understand mensuration and naturally wonder why there is not a formula for this purpose given.

The following simple deductions lead readily to such a formula.

Let A be the smallest angle of a triangle. Let the sides be denoted by a, b, c, and the area by $\triangle$. Also let $2s=a+b+c$.

Then, $\sin A=\dfrac{2\triangle}{bc}$, $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$, $2+\cos A=\dfrac{4bc+b^2+c^2-a^2}{2bc}$,

$2+\cos A=\dfrac{2s\,(s-a)+bc}{bc}$. Hence,

$$\frac{2\triangle}{2s\,(s-a)+bc}=\frac{\sin A}{2+\cos A}=\frac{A-\frac{1}{6}A^3}{3-\frac{1}{2}A^2},$$

omitting higher powers, or $\dfrac{2\triangle}{2s\,(s-a)+bc}=\dfrac{A}{3}$, in linear measure, $=\dfrac{\pi A}{3\times 180}$ in circular measure.

That is, $A=\dfrac{1080\triangle}{\pi[2s(s-a)+bc]}=\dfrac{344\triangle}{2s\,(s-a)+bc}$.

As a special case, let $a=20$, $b=51$, $c=65$. Then $\triangle=408$, $s=68$, $s-a=48$, $bc=3315$. Hence, $A=\frac{140352}{9843}=14.259°=14°\ 15'\ 3''$.

Similarly, $B=\dfrac{344\triangle}{2s\,(s-b)+ac}=\frac{140352}{3612}=38.857°=38°\ 51'\ 25''$.

The values by the tables are: $A=14°\ 15'$, $B=38°\ 52'\ 48''$.

If we have a right triangle, then $a^2+b^2=c^2$, and

$$A=\frac{172ab}{\frac{1}{2}\,(b^2+2bc+c^2-a^2)+bc}=\frac{172ab}{b^2+2bc}=\frac{172a}{b+2c}.$$

This value for the lesser angle of a right triangle, I am informed, is found in some work, but I cannot learn where or how it is deduced. I will greatly appreciate this information from some reader of the MONTHLY.

As an application of this formula, let $a=4476$, $b=7332.8$, $c=8590$. Then $A=\dfrac{769872}{24512.8}=31.407°=31°\ 24'\ 25''$. The value by the tables is $A=31°\ 24'$. The smaller the angle the less the error. Hence it is always best to find the small angle by this formula.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

298. Proposed by W. J. GREENSTREET, Marling School, Stroud, England.

Find an approximation to the difference between the sums of n harmonic and n arithmetic means between a and b, when a is very nearly equal to b.

Solution by A. H. HOLMES, Brunswick, Me., and J. SCHEFFER, A. M., Hagerstown, Md.

n arithmetic means between a and b would be:

$$\frac{an+b}{n+1},\quad \frac{a(n-1)+2b}{n+1},\quad \frac{a(n-2)+3b}{n+1},\ \ldots,\ \frac{a+nb}{n+1}.$$

Putting S_1 for the sum of these terms, we find $S_1=\dfrac{(a+b)n}{2}$.

n harmonic means between a and b would be:

$$\frac{ab(n+1)}{a(n+1)+(b-a)},\quad \frac{ab(n+1)}{a(n+1)+2(b-a)},\ \ldots,\ \frac{ab(n+1)}{a(n+1)+n(b-a)}.$$

Putting S_2 for the sum of these n terms we find, disregarding powers of $(b-a)$ above the third since a nearly equals b,

$$S_2=nb\left[1-\frac{b-a}{2a}+\frac{(2n+1)(b-a)^2}{6a^2(n+1)}-\frac{n(b-a)^3}{4a^2(n+1)}\right].$$

$$\therefore S_1-S_2=n\left[\frac{a+b}{2}-b\left(1-\frac{b-a}{2a}+\frac{(2n+1)(b-a)^2}{6a^2(n+1)}-\frac{n(b-a)^3}{4a^2(n+1)}\right)\right]$$

$$=n\left[\frac{a-b}{2}+\frac{b}{a}\left(\frac{b-a}{2}\right)-\frac{b(2n+1)(b-a)^2}{6a^2(n+1)}-\frac{nb(b-a)^3}{4a^2(n+1)}\right]$$

$$=\frac{n}{2a}(a-b)^2,\ \text{nearly.}$$

Also solved by G. B. M. Zerr, T. I. Wodo, and H. V. Spunar.

299. Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.

The sides of a triangle and the area are in arithmetical progression. Find their values, and show there is only *one* solution in rational integers.

Solution by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

To avoid fractions, take the sides and area to be, in order, $2x$, $2x+2y$, $2x+4y$, and $2x+6y$; then $3(x+y)$=the half sum of the sides, from which we have

$$[3(x+y)(x+3y)(x+y)(x-y)]^{\frac{1}{2}}=2x+6y\ldots(1).$$

Throw off the radical and divide each side of equation (1) by $(x+3y)(x+y)^2$ and we have after reduction

$$3(x-y)/(x+3y)=4/(x+y)^2=m^2\ldots(2).$$

The least value of m for positive integral results=1. Therefore $x=2-y$, and the sides and area in order are, $4-2y$, 4, $4+2y$, and $4+4y$.

The least value of y for positive integral results$=\frac{1}{2}$. Therefore, the sides and area are in order 3, 4, 5, and 6.

The triangle is a right triangle; and there are an indefinite number of similar triangles; integral or fractional multiples; but there is but *one* solution.

Also solved by George W. Hartwell, T. I. Wodo, G. B. M. Zerr, M. V. Spunar, A. H. Holmes, J. Scheffer, and J. M. Arnold.

172. Proposed by DR. L. E. DICKSON, The University of Chicago.

Without solving the algebraically solvable quintic, $y^5+py^3+\frac{1}{5}p^2y+r=0$, prove that it is irreducible in the domain of rationality (p, r).

Solution by H. S. VANDIVER, Bala, Pa.

Put the function in the form

$$5y^5+5py^3+p^2y+5r.$$

If the original function is reducible in domain (p, r) this function is also. The assumption that it is reducible in domain (p, r) is equivalent to the assumption that it can be expressed as the product of two factors:

$$x^n+\alpha_1x^{n-1}+\ldots+\alpha_n,$$
$$5x^{5-n}+\beta_1x^{4-n}+\ldots+\beta_{5-n},$$

where the α's and β's are rational functions in p and r. By Theorem VI, page 79, Vol. I, of Weber's *Algebra*, French edition, they may also be considered integral. The form of the factors shows that the function remains

reducible for all finite values of p and r. Let $p=5$, $r=5$. Then, ignoring constant factor 5, y^5+5y^3+5y+5 is reducible. But this is irreducible by the well known theorem of Eisenstein (Weber, l. c., p. 702).

The irreducibility may also be proved by setting $p=0$, $r=2$, whence the function y^5+2, which is irreducible by the theorem in Dickson's *Theory of Algebraic Equations*, p. 77, §90.

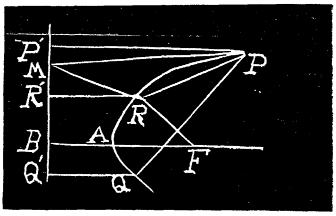
GEOMETRY.

332. Proposed by DAVID F. KELLEY, New York, N. Y.

To find the area of a parabolic sector, by a hitherto unpublished method.

Solution by the PROPOSER.

Let a secant meet a parabola in the points P and Q. Join the points P and Q to F which is the focus of the parabola. Let fall perpendiculars from P and Q on the directrix, BC, meeting it in the points P' and Q', respectively. Let R be any other point on the curve between P and Q. Join P and R, and let fall RR' perpendicular to directrix BC, and meeting it in the point R'. Bisect $P'R'$ in M, and join M and R, and M and P. By a well known geometrical theorem, area of $\triangle PMR=\frac{1}{2}$ quadrilateral $PP'R'R$. Let R' move indefinitely near to P', then, in the limit, $MR=R'R=FR$, and $NP=P'P=FP$. Therefore, in the limit, $\triangle PRF=\triangle PMR=\frac{1}{2}$ quadrilateral $PP'R'R$. Hence, it readily follows that space $FPRAQ=\frac{1}{2}$ space $PRAQQ'P'$. Hence, if $O=$space $PRAQQ'P'$, and $I=$space $PRAQ$, and $\triangle'=$area $\triangle FPQ$, and $k=$area of quadrilateral $PQQ'P'$, we have the following two equations connecting O and I:



$$\triangle'+I=\tfrac{1}{2}O,\quad O+I=k.$$

In particular, when PQ is perpendicular to AB, if x and y be coordinates of P, we have $(a-x)y+I=\frac{1}{2}O$. $I+O=2(a+x)y$, and solving for I we get $I=4xy/3$.

Again, since, in the limit, $\triangle FPR=\triangle MPR$, it follows that if perpendiculars be let fall from P' and F on the tangent to the parabola at P, then these perpendiculars are equal, and hence it is readily seen since $FP=PP'$ that the line joining F and P' is bisected by the tangent at P, and is at right angles to it.

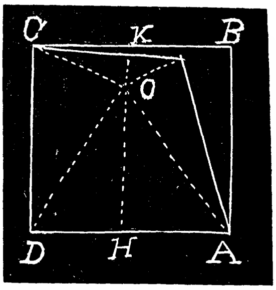
Also solved by G. B. M. Zerr, and H. V. Spunar.

333. Proposed by J. B. MORRELL, Boulder, Colorado.

Exhibit the fallacious argument to prove that a right-angle is equal to an angle which is less than a right-angle.

Solution by the PROPOSER.

Let $ABCD$ be a rectangle. From A draw a line AE inside the rectangle equal to AB or DC and making an acute angle with AB as indicated in the diagram.



Bisect AD in H and through H draw HO at right angles to DA. Bisect CE in K and through K draw KO at right angles to CE. Since CB and CE are not parallel the lines HO and KO will meet (say) at O. Join OA, OE, OC, and OD.

Now $OA=OD$ and therefore angle ODA equals angle OAD. OC equals OE, and by construction AE equals CD, therefore triangle COD equals triangle EOA and the homologous angles ODC and OAE are equal.

Now we have ODC equal to OAE, ODA equal to OAD, therefore $ODC+ODA$ equals $OAE+OAD$, or CDA is equal to EAD. But CDA is a right angle and EAD is less than a right angle, therefore the result is impossible.

334. Proposed by J. O. MAHONEY, B. E., M. Sc., Central High School, Dallas, Texas.

Through any point P in the plane of the triangle ABC, draw a line that shall divide the perimeter of the triangle into two equal parts.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

This problem reduces to the following: Through a given point in the plane of a given angle to draw a straight line which cuts off from the sides of the angle two intercepts the sum of which is equal to a given line, viz., the half-sum of the three sides of the triangle.

Let AX and AY be the sides of the angle; P be the given point. Draw $PD=n$, parallel to AY, and put $AD=m$. Denote the semi-perimeter of the triangle by s. Let MPN be the required line, M a point in AY and N in a point AX. Then $AM:n=AN:AN-m$, and $AM+AN=s$. From these two relations we find the quadratic

$$AN^2-(s-n+m)AN=-ms,$$

which admits of an easy geometric construction. Since this equation furnishes two values, and the three angles of a triangle admit of three combinations of two at a time, there are really six solutions of the problem.

Also solved by G. B. M. Zerr, H. V. Spunar, and C. N. Schmall.

CALCULUS.

261. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that $\sum_{x=1}^{x=\infty}\frac{1}{a+2bx^2+cx^4}=\frac{\pi}{\sqrt{[8ac(\sqrt{ac}+b)]}}-\frac{1}{2a}$, where $ac>b^2$.

Solution by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

$$\sum_{x=1}^{x=\infty}\frac{1}{a+2bx^2+cx^4}=\int_1^\infty\frac{dx}{a+2bx^2+cx^4}=A.$$

As $ac>b^2$, $a+2bx^2+cx^4=(\sqrt{a}+2kx+x^2\sqrt{c})(\sqrt{a}-2kx+x^2\sqrt{c})$, where $k=\sqrt{\frac{\sqrt{(ac)}-b}{2}}$.

$$\therefore A=\frac{1}{4k\sqrt{a}}\int_1^\infty\frac{(x+2k)dx}{\sqrt{a}+2kx+x^2\sqrt{c}}-\frac{1}{4k\sqrt{a}}\int_1^\infty\frac{(x-2k)dx}{\sqrt{a}-2kx+x^2\sqrt{c}}$$

$$=\left[\frac{1}{8k\sqrt{a}}\log\frac{\sqrt{a}+2kx+x^2\sqrt{c}}{\sqrt{a}-2kx+x^2\sqrt{c}}+\frac{1}{4\sqrt{[a(b+k^2)]}}\tan^{-1}\frac{2x\sqrt{(b+k^2)}}{\sqrt{a}-x^2\sqrt{c}}\right]_1^\infty$$

$$=\tfrac{1}{2}\left[\frac{\pi}{\sqrt{ah}}-\frac{1}{a}\right],\text{ where } h=2\sqrt{(ac)}+b,\ =\frac{\pi}{\sqrt{[8ac(\sqrt{ac}+b)]}}-\frac{1}{2a}.$$

Also solved by G. B. M. Zerr.

262. Proposed by H. SCHAFFER, Fayetteville, Ark.

Prove that the circle is the only plane curve of constant curvature.

Solution by C. N. SCHMALL, New York City.

The expression for the curvature of a plane curve, $F(x, y)=0$, is

$$\frac{1}{R}=\frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}=c,\text{ say...}(1).$$

Put $\frac{dy}{dx}=z$. $\because\frac{d^2y}{dx^2}=\frac{dz}{dx}$, and (1) becomes $\frac{dz/dx}{(1+z^2)^{\frac{3}{2}}}=c$, whence

$$dx=\frac{dz}{c(1+z^2)^{\frac{3}{2}}},\text{ and, therefore, } x=\frac{1}{c}\int\frac{dz}{(1+z^2)^{\frac{3}{2}}}=\frac{1}{c}\cdot\frac{z}{\sqrt{(1+z^2)}}.$$

$\therefore c^2x^2=\frac{z^2}{1+z^2}$. $\therefore z^2=\frac{c^2x^2}{1-c^2x^2}$, $z=\frac{cx}{\sqrt{(1-c^2x^2)}}$; *i. e.*, $\frac{dy}{dx}=\frac{cx}{\sqrt{(1-c^2x^2)}}$.

$\therefore dy=\frac{cxdx}{\sqrt{(1-c^2x^2)}}$, and $y=c\int\frac{xdx}{\sqrt{(1-c^2x^2)}}=c.\left(-\frac{1}{c^2}\sqrt{(1-c^2x^2)}\right)$

$=-\frac{1}{c}\sqrt{(1-c^2x^2)}$. Squaring, $y^2=\frac{1}{c^2}(1-c^2x^2)$, or $c^2(x^2+y^2)=1$, the equation of a circle.

Also solved by G. B. M, Zerr, and V. M. Spunar.

ERRATUM. In problem 264, Calculus, the proposer evidently meant

$$\left(\frac{d^2\phi}{d\psi^2}\right)^2 \text{ instead of } \left(\frac{d^2\phi}{d^2\psi}\right)^2.$$

MECHANICS.

215. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Determine the curve in a vertical plane along a chord of which a particle will slide under the force of gravity and the retardation of friction so that it will traverse the whole length of the chord in a time t which is independent of its direction as long as the upper end of the chord remains fixed. Discuss the result.

Solution by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Take the fixed end of the chord for the origin of the axes, that of x being horizontal, and that of y vertical. Let s denote the length of any chord drawn from the fixed point, and denote by θ the angle it makes with the horizon. Then, denoting the coefficient of friction by μ, we have $s=g(\sin\theta-\mu\cos\theta)\frac{t^2}{2}$, t being the time. If now, t is to be independent of θ, $\frac{s}{\sin\theta-\mu\cos\theta}$ must be a constant, say$=a$.

$$\therefore \frac{\sqrt{(x^2+y^2)}}{\frac{y}{\sqrt{(x^2+y^2)}}-\frac{\mu x}{\sqrt{(x^2+y^2)}}}=a, \text{ or, } x^2+y^2=a(y-\mu x), \text{ a circle, the}$$

coordinates of the center of which are $-\frac{1}{2}a\mu$ and $\frac{1}{2}a$, and radius $=\frac{a}{2}\sqrt{(1+\mu^2)}$.

Also solved by G. B. M. Zerr.

215. Proposed by HENRY WRITT, Genoa Junction, Wisconsin.

Suppose two centers of attractive forces A and B having a ratio 1 : 330,000, and influence reducing as the second power of the distance, *i. e.*,

R^{-2}. Then there is a point, P, on the line joining A and B, where $\frac{1}{AP^2}=\frac{330,000}{BP^2}$, or 1 : 575, nearly. At this point the attractions are equal but opposite in direction along AB. It is proposed to find the surface through the point P which is the locus of the direction of the resultant of the two forces directed towards A and B, *i. e.*, the locus of the diagonals of the minimum parallelogram of forces.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

We are to find a surface such that every point on this surface is equally attracted by both centers of force.

Let A be the origin; (x, y, z) the coordinates of any point on the surface sought; $AB=a$.

Then, $\frac{1}{x^2+y^2+z^2}=\frac{330000}{(a-x)^2+y^2+z^2}$.

$\therefore 329999(x^2+y^2+z^2)=a^2-2ax$ is the required surface.

This is a sphere radius $\frac{100a\sqrt{33}}{329999}$, with center distant $\frac{a}{329999}$ from A.

When $y=z=0$, $x=\frac{a}{1\pm100\sqrt{33}}=\frac{a(1\mp100\sqrt{33})}{329999}$.

$\therefore P$ divides a in the ratio 1 : 575, nearly.

217. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Given, the mean distance from earth to sun, 1.49×10^{15} centimeters; radius of the earth, 6.37×10^8 centimeters; velocity of the earth in its orbit, 2.96×10^6 centimeters per second; velocity of rotation of a point on the equator, 4.63×10^4 centimeters per second; mass of the earth, 6.14×10^{27} grams; find (1) the total energy of the earth in ergs; (2) the angular velocity of the earth on its axis; and (3) the angular velocity of the earth around the sun.

Solution by the PROPOSER.

Let m=earth's mass, v=its velocity of rotation, V=its velocity of translation.

(1) Energy of translation$=\frac{1}{2}mV^2$. Energy of rotation$=\frac{1}{2}I\omega^2=\frac{1}{5}mv^2$.

$$\text{Total energy}=m\left(\frac{V^2}{2}+\frac{v^2}{5}\right)=6.14\times10^{27}(4.3808\times10^{12}+4.2874\times10^8)$$

$$=269\times10^{38}\text{ dynes.}$$

(2) $r\omega=v$, or $\omega=v/r=\frac{4.63\times10^4}{6.37\times10^8}=.0000727$.

(3) $RU=V$, or $U=V/R=\frac{2.96\times10^6}{1.49\times10^{15}}=\frac{2}{10^9}=.000000002$.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

151. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

In the recurring series $u_0=3$, $u_1=0$, $u_2=2$, ..., where the scale of relation is $u_{n+3}=n_{n+1}+u_n$, prove that u_p is always divisible by p when p is prime. Is the converse true?

Solution by DR. L. E. DICKSON, The University of Chicago.

In this problem, we have $u_k=S_k$ for every k, where S_k denotes the sum of the kth powers of the roots of $x^3-x-1=0$. By the formula of Girard (often attributed to Waring),

$$(1) \qquad S_n=n\,\Sigma\frac{(m+l-1)!}{m!\,l!}\quad\left(\begin{array}{l}\text{summed for all integers} \geqq 0\\ \text{for which } 2m+3l=n.\end{array}\right)$$

Hence, for n a prime number, $u_n=S_n$ is divisible by n.

The problem admits of a wide generalization. Given any integers m, p_1, ..., p_m, with m positive, the recursion formula

$$(2) \qquad z_{x+m}+p_1z_{x+m-1}+p_2z_{x+m-2}+\ldots+p_mz_x=0$$

has the solution* $z_y=\sum_{i=1}^{m}C_ia_i^x$, in which the C's are arbitrary, while a_1, ..., a_m are the roots of

$$(3) \qquad a^m+p_1a^{m-1}+p_2a^{m-2}+\ldots+p_m=0.$$

The C_i may be expressed in terms of z_0, z_1, ..., z_{m-1}, and conversely. For suitably chosen (integral) values of the latter, we may make $C_1=1$, ..., $C_m=1$. Hence when relation (2) is arbitrarily assigned, we may construct an infinite series of integers z_0, z_1, ..., for which the recursion formula is (2) and such that

$$(4) \qquad z_k=S_k=\text{sum of } k\text{th powers of roots of (3)}.$$

Then z_p is given by Girard's formula

$$S_p=p\,\Sigma\frac{(-1)^{\lambda_1+\ldots+\lambda_m}(\lambda_1+\ldots+\lambda_m-1)!}{\lambda_1!\,\lambda_2!\ldots\lambda_m!}p_1^{\lambda_1}\ldots p_m^{\lambda_m},$$

summed for all integers $\lambda_i\geqq 0$ for which $\lambda_1+2\lambda_2+\ldots+m\lambda_m=p$. *Hence, for* $p_1=0$, *and p a prime*, z_p is divisible by p.

The only solution of the a's are distinct (*Encyclopaedie Mathematik*, Vol. 1, p. 934).

AVERAGE AND PROBABILITY.

195. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue Philadelphia, Pa.

A random diameter is drawn in a given circle. Find the chance that it intersects, (1) a random chord; (2) a random chord through a random point; and (3) a chord through two random points.

Solution by the PROPOSER.

Let O be the center of the given circle, AB the random chord, M the one random point, and N the second random point.

Let $OA=r$, $AM=x$, $\angle AOH=\theta$. For the second point, let $MN=y$ and let μ=angle AB makes with some fixed line

Then $AH=r\sin\theta$; an element of the circle at M is $r\sin\theta\, d\theta\, dx$; at n, $d\mu\, ydy$. The limits of θ are 0 and $\frac{1}{2}\pi$; of x, 0 and $2r\sin\theta=x'$; of y, 0 and x and doubled. Then the required chance is

$$(1)\quad p=\frac{\int_0^{\frac{1}{2}\pi}4\theta\, d\theta}{\int_0^{\frac{1}{2}\pi}2\pi\, d\theta}=\frac{4}{\pi^2}\int_0^{\frac{1}{2}\pi}\theta\, d\theta=\tfrac{1}{2}.$$

$$(2)\quad p=\frac{\int_0^{\frac{1}{2}\pi}\int_0^{x'}4r\theta\sin\theta\, d\theta\, dx}{\int_0^{\frac{1}{2}\pi}\int_0^{x'}2\pi r\sin\theta\, d\theta\, dx}=\frac{4}{\pi^2 r}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\theta\sin\theta\, d\theta\, dx$$

$$=\frac{8}{\pi^2}\int_0^{\frac{1}{2}\pi}\theta\sin^2\theta\, d\theta=\tfrac{1}{2}+\frac{2}{\pi^2}.$$

$$(3)\quad p=\frac{\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_0^{x}\int_0^{2\pi}4r\theta\sin\theta\, d\theta\, dx\, d\mu\, ydy}{\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_0^{x}\int_0^{2\pi}2\pi r\sin\theta\, d\theta\, dx\, d\mu\, ydy}$$

$$=\frac{4}{\pi^3 r^3}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_0^{x}\int_0^{2\pi}\theta y\sin\theta\, d\theta\, dx\, dy\, d\mu$$

$$=\frac{4}{\pi^2 r^3}\int_0^{\frac{1}{2}\pi}\int_0^{x'}x^2\theta\sin\theta\, d\theta\, dx=\frac{32}{3\pi^2}\int_0^{\frac{1}{2}\pi}\theta\sin^4\theta\, d\theta=\tfrac{1}{2}+\frac{8}{3\pi^2}.$$

If a random chord is drawn instead of a random diameter, we let $\angle COK=\phi$, $\angle AOC=\psi$. The limits of ϕ are 0 and θ and doubled; of ψ, 0 and 2ϕ and doubled.

$$\therefore (1)\quad p=\frac{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi} d\,\theta\, d\,\phi\, d\,\psi}{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{\pi} d\,\theta\, d\,\phi\, d\,\psi}=\frac{8}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi} d\,\theta\, d\,\phi\, d\,\psi$$

$$=\frac{16}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta} \phi\, d\,\theta\, d\,\phi=\frac{8}{\pi^3}\int_0^{\frac{1}{2}\pi} \theta^2 d\,\theta=\tfrac{1}{3} \text{ (same as 191).}$$

$$(2)\quad p=\frac{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi}\int_0^{x'} r\sin\theta\, d\,\theta\, d\,\phi\, d\,\psi\, dx}{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{\pi}\int_0^{x'} r\sin\theta\, d\,\theta\, d\,\phi\, d\,\psi\, dx}$$

$$=\frac{8}{\pi\, r(\pi^2+4)}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi}\int_0^{x'}\sin\theta\, d\,\theta\, d\,\phi\, d\,\psi\, dx$$

$$=\frac{16}{\pi(\pi^2+4)}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi}\sin^2\theta\, d\,\theta\, d\,\phi\, d\,\psi=\frac{16}{\pi(\pi^2+4)}\int_0^{\frac{1}{2}\pi}\theta^2\sin^2\theta\, d\,\theta$$

$$=\frac{\pi^2+6}{3(\pi^2+4)}=\tfrac{1}{3}+\frac{2}{3(\pi^2+4)}.$$

$$(3)\quad p=\frac{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi}\int_0^{x'}\int_0^{x} yr\sin\theta\, d\,\theta\, d\,\phi\, d\,\psi\, dx\, dy}{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{\pi}\int_0^{x'}\int_0^{x} yr\sin\theta\, d\,\theta\, d\,\phi\, d\,\psi\, dx\, dy}$$

$$=\frac{48}{\pi\, r^3(3\pi^2+16)}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi}\int_0^{x'}\int_0^{x} y\sin\theta\, d\,\theta\, d\,\phi\, d\,\psi\, dx\, dy$$

$$=\frac{64}{\pi(3\pi^2+16)}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi}\sin^4\theta\, d\,\theta\, d\,\phi\, d\,\psi=\frac{64}{\pi(3\pi^2+16)}\int_0^{\frac{1}{2}\pi}\theta^2\sin^4\theta\, d\,\theta$$

$$=\frac{\pi^2+9}{3\pi^2+16}=\tfrac{1}{3}+\frac{11}{3(3\pi^2+16)}.$$

This exhibits the reason for the $\frac{1}{2}$ given by some contributors to 191 instead of the correct value $\frac{1}{3}$.

MISCELLANEOUS.

173. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

If n is odd, prove the following: $\pm 1=[(-1)^{1/n}+(-1)^{-(1/n)}][(-1)^{2/n}+(-1)^{-(2/n)}][(-1)^{3/n}+(-1)^{-(3/n)}]\ldots[(-1)^{(n-1)/2n}+(-1)^{-(n-1)/2n}]$
$\pm\sqrt{n}(-1)^{(n-1)/4}=[(-1)^{1/n}-(-1)^{-(1/n)}][-(-1)^{2/n}-(-1)^{-(2/n)}][(-1)^{3/n}-(-1)^{-(3/n)}]\ldots[(-1)^{(n-1)/2n}-(-1)^{-(n-1)/2n}]$.

Solution by the PROPOSER.

Delete the factor $(-1)^{(n-1)/4}$ from the first member of the second expression. To resolve into factors the expression $x^n-1=0$. When n is odd,

$$x^n=1=\cos 2m\pi\pm i\sin 2m\pi=(-1)^{\pm 2m},$$

since $\cos 2m\pi=\frac{1}{2}[(-1)^{2m}+(-1)^{-2m}]$, $\sin 2m\pi=\dfrac{1}{2i}[(-1)^{2m}-(-1)^{-2m}]$.

Giving m the values 0, 1, 2, 3, etc.,

$$x=(-1)^{\pm 0},\ (-1)^{\pm(2/n)},\ (-1)^{\pm(4/n)},\ \ldots,\ (-1)^{\pm(n-1)/n}.$$

$\therefore$ The roots are 1, $(-1)^{\pm(2/n)}$, $(-1)^{\pm(4/n)}$, ..., $(-1)^{\pm(n-1)/n}$, and the factors are $(x-1)$, $[x-(-1)^{2/n}][x-(-1)^{-(2/n)}]$, $[x-(-1)^{4/n}][x-(-1)^{-(4/n)}]$, ..., $[x-(-1)^{(n-1)/n}][x-(-1)^{-(n-1)/n}]$.

$\therefore x^n-1=(x-1)[x^2-x(-1)^{2/n}-x(-1)^{-(2/n)}+1][x^2-x(-1)^{4/n}-x(-1)^{-(4/n)}+1]$, ..., $[x^2-x(-1)^{(n-1)/n}-x(-1)^{-(n-1)/n}+1]$.

If $x=-1$, $(x^n-1)/(x-1)=1$.

$\therefore 1=[2+(-1)^{2/n}+(-1)^{-(2/n)}][2+(-1)^{4/n}+(-1)^{-(4/n)}]$, ..., $[2+(-1)^{(n-1)/n}+(-1)^{-(n-1)/n}]$,
$=[(-1)^{1/n}+(-1)^{-(1/n)}]^2[(-1)^{2/n}+(-1)^{-(2/n)}]^2$, ..., $[(-1)^{(n-1)/2n}+(-1)^{-(n-1)/2n}]^2$.

$\therefore \pm 1=[(-1)^{1/n}+(-1)^{-(1\div n)}][(-1)^{2\div n}+(-1)^{-(2\div n)}][(-))^{3\div n}+(-1)^{-(3\div n)}]\ldots[(-1)^{(n-1)\div 2n}+(-1)^{-(n-1)\div 2n}]$.

If $x=1$, $(x^n-1)/(x-1)=x^{n-1}+x^{n-2}+\ldots+x+1=n$.

$\therefore n=[(-1)^{1\div n}-(-1)^{-(1\div n)}]^2[(-1)^{2\div n}-(-1)^{-(2\div n)}]^2$, ..., $[(-1)^{(n-1)\div 2n}-(-1)^{-(n-1)\div 2n}]^2$.

$\therefore \pm\sqrt{n}=[(-1)^{1\div n}-(-1)^{-(1\div n)}][(-1)^{2\div n}-(-1)^{-(2\div n)}][(-1)^{3\div n}-(-1)^{-(3\div n)}]\ldots[(-1)^{(n-1)\div 2n}-(-1)^{-(n-1)\div 2n}]$.

175. Proposed by PROFESSOR R. D. CARMICHAEL, Anniston, Ala.

If x and z are connected by the relation $z=zf(x)+x\phi(z)$, find the value of $f(z)$ in the form of a power series in x with constant coefficients. In particular, give such a value of z when $z=z\sin x+x\cos z$.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Write the expression in the form $[1-f(x)]x^{-1}=\phi(z)z^{-1}$.
Expand both sides and find z by reversion of series $\dfrac{1-\sin x}{x}=\dfrac{\cos z}{z}$.

$$\therefore\ \frac{1}{x}-1+\frac{x^2}{3!}-\frac{x^4}{5!}+\frac{x^6}{7!}-\ldots=\frac{1}{z}-\frac{z}{2!}+\frac{z^3}{4!}-\frac{z^5}{6!}+\ldots$$

$$\therefore az+\frac{z^2}{2!}-\frac{z^4}{4!}+\frac{z^6}{6!}-\ldots=1,\text{ where } a=\frac{1}{x}-1+\frac{x^2}{3\,!}-\frac{x^4}{5!}+\ldots$$

By reversion of series,

$$z=\frac{1}{a}-\frac{1}{2a^3}+\frac{1}{2a^5}-\frac{15+a^2}{24a^7}+\frac{7+a^2}{8a^9}-\ldots,\text{ where } a=\frac{1-\sin x}{x}.$$

$$\therefore z=\frac{x}{(1-\sin x)}-\frac{x^3}{2\,(1-\sin x)^3}+\frac{11x^5}{24\,(1-\sin x)^5}-\frac{x^7}{2(1-\sin x)^7}+\ldots$$

$$=x+x^2+\tfrac{1}{2}x^3-\tfrac{2}{3}x^4-\tfrac{15}{8}x^5-\tfrac{39}{20}x^6-\ldots$$

II. Solution by V. M. SPUNAR, East Pittsburg, Pa.

$z=zf(x)+x\,\phi(z)$ transformed gives $f(z)=\phi(z)/z=x^{-1}[1-f(x)]$.
Expanding $f(x)$ by means of McLaurin's Theorem,

$$f(x)=f(0)+\frac{x}{1\,!}f'(0)+\frac{x^2}{2\,!}f''(0)+\frac{x^3}{3\,!}f'''(0)+\ldots+\frac{x^n}{n\,!}f^{(n)}(0).$$

$$\therefore f(z)=x^{-1}-x^{-1}f(0)-\frac{1}{1\,!}f'(0)-\frac{x}{2\,!}f''(0)-\frac{x^2}{2\,!}f'''(0)-\ldots-\frac{x^{n-1}}{n\,!}f^{(n)}(0).$$

In particular, $z=z\sin x+x\cos z$; $f(z)=\cos z/z=x^{-1}(1-\sin x)$. Since

$$f(x)=\sin x=\frac{1}{1\,!}-\frac{x^3}{3\,!}+\frac{x^5}{5\,!}-\frac{x^7}{7\,!}+\frac{x^9}{9\,!}-\frac{x^{11}}{11!}\ldots\quad -\infty<x<+\infty.$$

$$\therefore f(z)=\frac{\cos z}{z}=x^{-1}-1+\frac{x^2}{3\,!}-\frac{x^4}{5\,!}+\frac{x^6}{7\,!}-\frac{x^8}{8\,!}+\frac{x^{10}}{11!}\ldots+(-1)^{n+1}\frac{x^{n-1}}{n!}.$$

PROBLEMS FOR SOLUTION.

ALGEBRA.

306. Proposed by J. C. CORBIN, Pine Bluff, Ark.

Muir gives the following problem:

$$\text{Prove: } \begin{vmatrix} 1 & a & a & a^2 \\ 1 & b & b & b^2 \\ 1 & c & c' & cc' \\ 1 & d & d' & dd' \end{vmatrix} = (a-b) \begin{vmatrix} 1 & ab & a+b \\ 1 & cd' & c+d' \\ 1 & c'd & c'+d \end{vmatrix}$$

which, of course, can be solved by finding the terms of both determinants. Is there any method of changing from one form to the other which is direct?

307. Proposed by J. SCHEFFER, Hagerstown, Md.

If $y^x=2$ and $x^y=3$, find x and y.

GEOMETRY.

339. Proposed by G. E. BROCKWAY, Boston, Mass.

Of all triangles that can be inscribed in a given triangle, that formed by joining the feet of the altitudes has the minimum perimeter. Prove by means of the straight line and circle.

340. Proposed by J. H. MEYERS, S. J., Sacred Heart College, Augusta, Ga.

Given trapezoid $ABCD$. Prolong AB and CD, the non-parallel sides, to meet in E. On AE as diameter construct semi-circle $ALGE$. With BE as radius construct arc BG. Draw GH perpendicular to AE. Bisect AH at K. Erect KL perpendicular to AE. Construct arc LM with LE as radius. Draw MN perpendicular to DC. Prove that MN bisects the trapezoid $ABCD$, angles ADC and BCD being right angles.

CALCULUS.

265. Proposed by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

Find two curves which possess the property that the tangents TP and TQ to the inner one always make equal angles with the tangent TT' to the outer.

266. Proposed by C. N. SCHMALL, New York City.

Show that the nth derivative of the fraction u/v can be expressed in the form of a determinant, u and v being functions of x.

MECHANICS.

221. Proposed by W. J. GREENSTREET, Stroud, England.

Two smooth intersecting planes are each at 45° to the horizon. Between them lies a cylinder of elliptic cross section. Find the position of equilibrium.

222. **Proposed by W. J. GREENSTREET, Stroud, England.**

Find the maximum angle of inclination to the line of greatest slope of a uniform rod resting on a rough inclined plane and capable of turning freely round a point on it.

BOOKS.

Analytical Geometry for Colleges, Universities, and Technical Schools. By E. W. Nichols, Professor of Mathematics in the Virginia Military Institute. Revised edition. 8vo. Cloth sides and leather back. xi+282 pp. Price, $1.25. Boston and Chicago: D. C. Heath & Co.

Professor Nichols' Analytical Geometry has enjoyed an extended popularity for some years. In the revision of the work, no important changes have been made but several minor changes have been made in order to bring the work in close relation with modern views. This edition is put in pocket form for greater convenience. B. F. F.

Elements of Physics. By George A. Hoadley, C. E., Sc. D., Professor of Physics in Swarthmore College. 8vo. Cloth, 464 pages. Price, $1.20. New York and Chicago: American Book Co.

In this book, the fundamental facts of the subject are presented in logical order and in clear and simple language. Much attention is given to the application of the principles of Physics to every day experiences. The illustrations are good and the book, from the mechanical point of view, is neatly gotten up. B. F. F.

An Elementary Treatise on the Differential Calculus Founded on the Method of Rates. By William Woolsey Johnson, Professor of Mathematics at the U. S. Naval Academy, Annapolis, Md. Abridged edition. Small 8vo. Cloth, x+191 pages; 52 figures. Price, $1.50. New York: John Wiley & Sons.

This book is in great part an abridgment of the author's larger treatise on the same subject. However, the earlier part is entirely revised and the simpler method than the functional method is used, in establishing the principal formulae of differentiation. Throughout the book graphical methods, notably in the case of trigonometric functions, are given preference. Illustrative examples occur in large numbers, and an extensive list of examples with answers follow the several sections. The subject of the Differential Calculus is presented in the work in a very teachable form. B. F. F.

Traité de Mathématiques Générales a l'Usage des Chemistes, Physiciens, Ingénieurs, et des Facultés des Sciences, avec Préface de G. Darboux, Secretaire perpetuel de L'Académie des Sciences. 8vo. Paper cover, x+440 pages. Price, paper cover, 9 fr.; cloth cover 10.50 fr. Paris: A. Hermann & Fils.

As its name indicates, the work is an elementary exposition of general mathematics. It is divided into four parts. The first part treats of Algebra; the second, Analytical Geometry; the third, Analysis; and the fourth, Mechanics. The book is written in clear style and is free from the severer analysis of higher mathematics. It will prove to be of much value to the general student of mathematics as well as to the student of engineering. F.

Plane and Solid Geometry. By Elmer A. Lyman, Professor of Mathematics in the Michigan State Normal College; Ypsilanti, Mich. 8vo. Cloth sides and leather back, 340 pages. Price, $1.75. New York and Chicago: American Book Co.

This book adds human interest to the study of Geometry by introducing now and then a brief historical note. A portrait of Euclid is the frontispiece. B. F. F.

The Foundations of Mathematics. A contribution to the philosophy of Geometry. By Paul Carus. 8vo. Red cloth, gilt top, iv+141 pages. Price, 75 cents net. Chicago: The Open Court Publishing Co.

This work is a very notable and valuable addition to the list of the Open Court Mathematical publications. The author, who is not a mathematician, but a philosopher, has given a very clear exposition of a subject about which there is general misunderstanding and contention. Dr. Carus is a lucid writer, and his discussion of the "Parallel Theorem," the "Fourth Dimension" and other equally interesting subjects is put in such a non-technical form as to be easily understood by the non-mathematical reader. In his Epilogue, Dr. Carus brings out strongly the analogy between mathematics and religion, the ultimate and unchangeable form of being and God. A very interesting and readable book for all classes of readers. B. F. F.

Graded Exercises in Phonography. By William Lincoln Anderson. 137 pages. Price 50 cents. Boston and Chicago: Ginn & Co.

This is an exercise book containing the writing exercises of "American Phonography," by William L. Anderson. B. F. F.

Algebra for Secondary Schools. By Charles Davison, Sc. D., Mathematical Master at King Edward's High School, Birmingham, England. 8vo. Cloth, viii+623 pages. Cambridge: The University Press. New York: G. P. Putnam's Sons.

This book is well adapted for use in all secondary schools. Three chapters are devoted to graphs. The problems are numerous and well selected. While the subjects discussed are those commonly included in most text-books on this subject, a few, such as the remainder theorem and simpler partial fractions, are introduced at an earlier stage than usual. The book is well written and material well arranged. B. F. F.

Elementary Algebra. By C. H. French, M. S., and G. Osborn, M. S., Mathematical Masters at the Leys School, Cambridge, formerly Scholars of Emanuel College, Cambridge. 8vo. Cloth, xii+506 pages. Cambridge: The University Press. New York: G. P. Putnam's Sons.

This is a revised and enlarged form of a text which the authors say has met with gratifying success. While some changes have been made in accordance with modern methods, the authors have carefully retained the main distinctive feature of the original work, viz., simplicity of style. The examples are largely original but a number have been taken from examination papers set at Cambridge and elsewhere. In this work as in the previous one, the answers are put at the end of the volume. B. F. F.

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THE AMERICAN MATHEMATICAL MONTHLY.

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ON THE FACTORIZATION OF LARGE NUMBERS.

By PROFESSOR L. E. DICKSON, Ph. D., The University of Chicago.

1. In the study of a difficult problem, it is a decided handicap to be denied the useful information that accompanies a knowledge of the origin of the proposed problem. There is little interest and much labor in the factorization of numbers taken at random. The real desideratum is a method which is capable of making effective use of the information which can be derived from the origin of the proposed number, and of auxiliary tables at command. For example, we may be concerned with numbers of a given form such as $m^n \pm 1$, or with the eliminant* of a system of congruences under investigation. We shall here illustrate such a method by determining the composition, hitherto unknown, of two numbers each of eleven digits. The first of these is a, where

$$A=\tfrac{1}{27}(26^{13}+1)=937.6449.a, \quad a=15207498827.$$

The first two prime factors were obtained by Lt. Col. Cunningham by means of his tables of solutions of $y^n \pm 1 \equiv 0 (\text{mod. } q)$, for $n<16$, q prime and $<10^4$.

2. Let p be a prime factor of a. Applying Fermat's theorem, we have

$$26^{p-1} \equiv 1, \quad 26^{26} \equiv 1 (\text{mod. } p),$$

so that $p-1$ is an even multiple of 13. Thus $p=1+26n$. Now

$$a=1+26N, \quad N=584903801.$$

Let $a=pq$. Then $1 \equiv q (\text{mod. } 26)$, $q=1+26n_1$. Then $a=pq$ gives

$$N=n+n_1+26nn_1. \tag{1}$$

*Instances of rapid factorizations of numbers known to be true eliminants occur in the writer's paper, "On the last theorem of Fermat," *Quarterly Journal of Mathematics*, Vol. 40 (1908), p. 40.

But $N \equiv 1 \pmod{26}$. Hence $n+n_1=1+26l$, l an integer. Then

$$nn_1+l=M\equiv 22496300.$$

If l were odd, n and n_1 would both be odd, whereas $n+n_1$ is odd. Thus $l=2t$. Then $(n_1-n)^2=(n_1+n)^2-4nn_1$ has the value

$$Q=(1+52t)^2-4(M-2t).$$

Thus Q must be a square. But $M\equiv 2$, $Q\equiv t^2+t-1 \pmod{3}$. If $t^2+t\equiv 0$, Q would be a quadratic non-residue of 3. Hence t is not congruent to $0, -1 \pmod{3}$. Thus $t=3k+1$,

$$Q=(156k+53)^2+24k+8-4M. \tag{2}$$

Since $1+26n>10^4$, $n\geqq 385$, $n_1\leqq 58490$. On the hyperbola (1), $n+n_1$ is a minimum when $n=n_1$, viz., for n approximately $474a$, since $\sqrt{a}$ just exceeds 123300. Thus $l\geqq 365$, $k\geqq 60$. For the above limits the maximum value of $n+n_1$ is approximately $385+58490$; hence $l\leqq 2264$, $k\leqq 377$.

We form the residues of Q modulo r, where r is one of the primes 5, ..., 23, and require that Q be a quadratic residue* of r. Thus

$$\begin{aligned}
&Q\equiv k^2+2,\ k\equiv \pm 2 \pmod{5};\\
&Q\equiv 4[(k-2)^2+1],\ k\equiv 1, 2, 3 \pmod{7};\\
&Q\equiv 4[(k+2)^2-3],\ k\equiv 0, 3, 4, 7, 8, 10 \pmod{11};\\
&Q\equiv -2k+3,\ k\equiv 0, 1, 2, 3, 6, 8, 10 \pmod{13};\\
&Q\equiv 9[(k+2)^2+1],\ k\equiv 2, 3, 5, 8, 10, 11, 14, 15, 16 \pmod{17};\\
&Q\equiv 16[(k-5)^2+6],\ k\equiv 4, 5, 6, 11, 12, 14, 15, 17, 18 \pmod{19};\\
&Q\equiv 25(k^2-6),\ k\equiv \pm 1, \pm 3, \pm 8, \pm 9, \pm 10, \pm 11 \pmod{23}.
\end{aligned}$$

From the results for $r=5$, $r=7$, and $60\leqq k\leqq 377$, we have

$$k=35x_1+2,\ 35x_2+3,\ 35x_3+8,\ 35x_4+17,\ 35x_5+22,\ 35x_6+23\ (2\leqq x_i\leqq 10).$$

Then, modulo 11, $\frac{1}{2}k\equiv x+1, 7, 4, 3, 0, 6$. But $\frac{1}{2}k\equiv 0, 2, 4, 5, 7, 9$. Thus†

$$\begin{aligned}
&x_1=3, 4, 6, 8, 10;\quad x_2=2, 4, 6, 8, 9;\quad x_3=3, 5, 7, 9;\\
&x_4=2, 4, 6, 8, 10;\quad x_5=2, 4, 5, 7, 9;\quad x_6=3, 5, 7, 9, 10.
\end{aligned}$$

Modulo 13, $3k\equiv x+6, 9, 11, 12, 1, 4$. But $3k\equiv 0, 3, 4, 5, 6, 9, 11$. Hence,

*These are given by the tables of indices in texts on the theory of numbers.

†The x_i are obtained by addition and suppressing positive residues other than 2, ..., 10.

$$x_1=3, 5, 7, 10;\quad x_2=2, 4, 7, 8, 9, 10;\quad x_3=2, 5, 6, 7, 8;$$
$$x_4=4, 5, 6, 7, 10;\quad x_5=2, 3, 4, 5, 8, 10;\quad x_6=2, 5, 7, 9.$$

For modulo 17, $35x_i\equiv x_i$. Hence

$$x_1=3, 6, 8, 9;\quad x_2=2, 5, 7, 8;\quad x_3=2, 3, 6, 7, 8;$$
$$x_4=2, 3, 5, 8, 10;\quad x_5=3, 5, 6, 9, 10;\quad x_6=2, 4, 5, 8, 9, 10.$$

The values of the x_i common to the three sets are

$$x_1=3;\quad x_2=2, 8;\quad x_3=7;\quad x_4=10;\quad x_5=5;\quad x_6=5, 9.$$

Modulus 19 excludes $x_2=2$, $k\equiv16$; $x_5=5$, $k\equiv7$; $x_6=5$, $k\equiv8$. Modulus 23 excludes $x_2=8$, $k\equiv7$; $x_3=7$, $k\equiv0$; $x_6=9$, $k\equiv-7$. Of the two remaining values,* $x_1=3$ gives $k=107$, $l=6k+2=644$, whence

$$Q=(13799)^2,\quad n_1-n=13799,\quad n_1+n=16745,\quad n_1=15272,$$
$$n=1473,\quad 1+26n=38299,\quad 1+26n_1=397073,$$

the two prime factors of a.

3. We next determine the composition of

$$b=31401724537=\frac{56^7-1}{56-1}=1+56N,\quad N\equiv\frac{56^6-1}{56-1}.$$

By Fermat's theorem, a prime factor p of b has the form $14k+1$ and hence $56l+1$, $+15$, $+29$, $+43$. The second and third forms are excluded by Legendre's table of divisors of quadratic forms, or as follows. If $p=56l+15$, 2 is a quadratic residue of p, and 7 a non-residue, in view of the reciprocity law. Thus 14 is a non-residue of p, contrary to $56^8\equiv56 \pmod{p}$.

If $b=pq$, then $q\equiv1 \pmod{14}$, $q=1+14k_1$. Thus

$$k+k_1+14kk_1=4N=4.57(56^4+56^2+1).$$

Hence $k+k_1\equiv4 \pmod{14}$, so that

$$(3)\qquad k+k_1=4+14h,\quad kk_1+h=16.57(56^3+56)+16.$$

First let $k=4l+3$, so that $p=56l+43$. By (3), $k_1+3\equiv2h$, $3k_1+h\equiv0 \pmod{4}$, whence $k_1\equiv h\equiv3 \pmod{4}$. Set $h=4t+3$, $k_1=4l_1+3$. Then by (3),

*For $x_4=10$, $k=367\equiv-10 \pmod{29}$, $Q\equiv3$, a quadratic non-residue of 29.

$$l+l_1=14t+10, \quad 4ll_1+43t=4.57(56^3+56)-29.$$

By the latter, $t\equiv 1 \pmod{4}$, $t=4c+1$. Hence

$$\tfrac{1}{4}(l_1-l)^2=S=16(7c+3)^2+43c+18-57(56^3+56).$$

Thus $S\equiv 3c+2 \pmod{8}$. If c is odd, $S\equiv 1$, $c\equiv 5 \pmod{8}$. If c is even, S must be a multiple of 4, whence $c\equiv 2 \pmod{4}$, $c=4m+2$, $\tfrac{1}{4}S\equiv 43m \pmod{32}$. If m is odd, then $m\equiv 3 \pmod{8}$. If m is even, then $m=4r$, and $\tfrac{1}{16}S\equiv 3r \pmod{8}$, so that either r is a multiple of 4 or $r\equiv 3 \pmod{8}$. Hence we have the cases

(4) $$c\equiv 5 \pmod{8}, \quad c=32w+14, \quad 64w+2, \quad 128w+50.$$

Modulo 81, S is the product of 16 by $S'=49c^2+70c+15$. In particular, $S'\equiv c^2+c \pmod{3}$. Thus $c\equiv 0$ or $2 \pmod{3}$. If c is a multiple of 3, S' must be a multiple of 9, so that $c=3+9d$, $S'\equiv 9(4d+2)$, mod. 81. Thus, $4d+2\equiv 0, 1, 4, 7 \pmod{9}$, $d\equiv 4, 2, 5, 8$. Next, if $c=2+3e$, $S'\equiv 6e \pmod{9}$, $e=3f$. Thus $S'\equiv 9(5f+3)$, mod. 81, $f\equiv 2, 3, 5, 8 \pmod{9}$. Hence

(5) $$c\equiv 20, 21, 29, 39, 47, 48, 74, 75 \pmod{81}.$$

$$\begin{aligned}
&S\equiv 4c^2+3,\ c\equiv \pm 2 \pmod{5}; \quad S\equiv c+1,\ c\equiv 0, 1, 3, 6 \pmod{7};\\
&S\equiv 16(5c^2+3),\ c\equiv 0,\ \pm 2,\ \pm 3 \pmod{11};\\
&S\equiv 4(c^2+1),\ c\equiv 0,\ \pm 3,\ \pm 4,\ \pm 5 \pmod{13};\\
&S\equiv 2[(c-4)^2-2],\ c\equiv 2, 3, 4, 5, 6, 10, 11, 14, 15 \pmod{17}.
\end{aligned}$$

By the tables cited, b has no factor $<10^4$. Thus, $1+14k\geqq 9999$, $1+14k_1\leqq 3140172$. Hence $k\geqq 714$, $k_1<224298$, $l\geqq 178$, $l_1<56074$. The sum of the latter gives the maximum $l+l_1$. Thus $t<4018$, $c\leqq 1004$. Since $\sqrt{b}$ just exceeds 177205, the approximate value for equal k's just exceeds 12657. Thus the equal l's just exceed 3164. Hence $t\geqq 451$, $c>112$.

For the first case under (4), we set $c=81x+20, \ldots, 75$, by (5). Thus

$$5\equiv x+4, 5, 5, 7, 7, 0, 2, 3; \quad x\equiv 1, 0, 0, 6, 6, 5, 3. 2 \pmod{8}.$$

The resulting values of c between 112 and 1005 are

525, 669, 749, 885, 965; 533, 237, 677; 453, 317.

The first five are excluded by mod. 5, the next three by mod. 11, the last two by mod. 7.

For $c=32w+14$, $4 \leq w \leq 30$ by the limits on c. By (5),

$$w \equiv 66, 23, 3, 59, 39, 77, 12, 50 (\text{mod. } 81),$$

respectively. Hence $w=23, 12$. But 23 is excluded mod. 5, and 12 mod. 17.

For $c=64w+2$, $2 \leq w \leq 15$. But $w \equiv 0, 4 (\text{mod. } 5)$, $w \equiv 0, 1, 2, 5, 8 (\text{mod. } 11)$. Hence $w=5$, $c=322 \equiv 79 (\text{mod. } 81)$, and is excluded by (5).

For $c=128w+50$, $1 \leq w \leq 7$. By (5), $w=3$, $c=434$, excluded mod. 5.

4. It remains to determine whether or not b has a factor $1+56n$. The complementary factor is of the form $1+56n_1$. Hence

$$n+n_1+56nn_1=N.$$

By inspection, $N \equiv 1 (\text{mod. } 56)$. Hence there is an integer l for which

$$n+n_1=56l+1, \quad nn_1+l=C=\frac{56^5-1}{56-1}=10013305,$$
$$(n_1-n)^2=S=(56l+1)^2+4l-4C.$$

Modulo 56, $S \equiv 4l-3$. Thus $S \equiv 1 (\text{mod. } 8)$, $l \equiv 1 (\text{mod. } 2)$, $l=2\lambda+1$. Also $S \equiv \lambda+1 (\text{mod. } 7)$, $\lambda \equiv 0, 1, 3, 6 (\text{mod. } 7)$. We have

$$S=112^2\lambda^2+8.1597\lambda-40049967.$$

Modulo 81, S is the product of $112^2 \equiv -11$ by $\sigma=\lambda^2+2\lambda+15$. The latter must be a quadratic residue of 81. In particular, $\lambda+1 \equiv \pm 1+3t$. Then $\sigma \equiv 6 \pm 6t (\text{mod. } 9)$; thus $\sigma \equiv 0 (\text{mod. } 9)$, $t \equiv \mp 1 (\text{mod, } 3)$, $\lambda+1=\mp 2+9f$. Then $\sigma \equiv 9(2 \mp 4f)$, mod. 81. Thus $2 \mp 4f$ is one of the quadratic residues 0, 1, 4, 7 of 9, whence $\pm f \equiv 5, 7, 4, 1 (\text{mod. } 9)$. Hence

$$\lambda \equiv 6, 19, 33, 37, 42, 46, 60, 73 (\text{mod. } 81).$$
$$4S \equiv (\lambda+2)^2-2, \quad \lambda \equiv 2, 4 (\text{mod. } 5);$$
$$S \equiv 4[(\lambda+2)^2+4], \quad \lambda \equiv 2, 5, 8, 9, 10 (\text{mod. } 11);$$
$$S \equiv 25[(\lambda-5)^2-3], \quad \lambda \equiv 0, 1, 3, 5, 7, 9, 10 (\text{mod. } 13);$$
$$8S \equiv (\lambda+2)^2-9, \quad \lambda \equiv \pm 1, -2, \pm 3, -5, 6, \pm 7 (\text{mod. } 17).$$

Since b has no factor $<10^4$, $n>178$, $n_1<56075$. The maximum $n+n_1$ is approximately 56253, whence $l<1005$, $\lambda<502$. The minimum $n+n_1$ is given by $n=n_1=3165-$. Hence $l \geq 113$, $\lambda \geq 56$. From the above residues moduli 81 and 5,

$$\lambda=405t+19, 37, 42, 87, 114, 127, 154, 199, 204, 222, 249, 262, 289, 357, 384, 397.$$

For the first three $t=1$; for the fourth $t=0, 1$; for the others $t=0$. Of the 17 resulting values of λ, 114, 222, 249, 289, 397, 424, 492 are excluded by mod. 7; then 127, 154, 199, 204, 447 are excluded by mod. 11; 262 and 357 by mod. 13; 87 and 442 by mod. 17; for the remaining value $\lambda=384$, $S\equiv 21$ (mod. 23), whereas 21 is a quadratic non-residue of 23.

Hence $b=\frac{1}{55}(56^7-1)$ *is a prime.*

While it is believed that the above work is accurate, having been carefully checked, it should be added that the same result was found by an earlier proof different as to details.

5. By the same method, I obtain the following results:

$$56^7+1=3.19.15737.1925393,$$
$$34^{17}+1=5.7.307.443.1531.28051.112643.4708729,$$
$$52^{13}+1=53.4057.21841.4328028093013,$$

all of the given factors being prime. That the last number of 13 digits is prime, I have verified by two proofs differing as to details. The factor 21841 was found by accident by Lt. Col. Cunningham. I ran across the factor 112643 of the second number in the manner explained in the *Quarterly Journal*, 1908, page 45; but the remaining two large factors were found by the present method.

6. In view of the interest in the numbers m^m-1 and their importance in connection with the last theorem of Fermat, it is desirable that some arithmetician should check the statement of E. Lucas (*American Journal of Mathematics*, Vol. 1, 1878, p. 294) that the large factors of 10 and 12 digits in $22^{11}\pm 1$ are actually primes. For a verification by the present method it is of the greatest help to know that there are no factors less than 10,000, in view of the tables by Lt. Col. Cunningham. The latter believes that Lucas intended to record his factors as primes; but that an uncertainty runs right through his factorizations as to the primality of the factors, no clue whatever being given as to how the primality was detected.

FACTORING IN A DOMAIN OF RATIONALITY.

By ELIZABETH R. BENNETT, The Universtty of Illinois.

If a series of symbols R_1, R_2, ... which are supposed to obey the ordinary laws of algebra, but are not necessarily thought of as representing numbers, are combined with respect to the four fundamental operations of arithmetic—addition, subtraction, multiplication, and division, division by zero being excluded, there result a series of expressions which are rational

with respect to these symbols. The totality of such expressions is called a domain of rationality and it is the smallest possible domain involving the symbols R_1, R_2, ...

If only one of these symbols as R_1 is involved in the combinations and if R_1 is a rational number different from zero, the domain of rational numbers including zero, positive and negative integers, and positive and negative rational fractions is obtained. This domain of rational numbers is included in every domain. The complex numbers of form $a+bi$, where a and b are rational numbers and $i=\sqrt{-1}$, also constitute a domain of rationality which includes the domain of rational numbers.

When we add or adjoin to a known domain a number β which does not already belong to it, the new set of numbers constitutes a domain, if we also add to it all numbers obtained from additions, subtractions, multiplications, and divisions involving β and all numbers of the original domain. The domain of the ordinary complex numbers already mentioned may be formed by the adjunction of i to the domain of rational numbers.

An algebraic integer is a root of the equation

$$x^n+a_1x^{n-1}+a_2x^{n-2}...+a_n=0$$

where a_1, a_2, ..., a_n are rational integers. An algebraic quadratic integer is a root of the above equation when $n=2$. All algebraic quadratic integers are of the form $x+y\sqrt{m}$ when $m\equiv 2$ or 3 (mod. 4), and of the form $x+y\dfrac{1+\sqrt{m}}{2}$ when $m\equiv 1$ (mod. 4). It is assumed that m is not divisible by any square greater than 1 and that x and y are integral.

The term integral domain is understood, as usual, to mean a set of integral elements which is invariant with respect to addition, subtraction, and multiplication; that is, any combinations of the numbers of the set by the operations mentioned yield again a number belonging to the set. An integral algebraic domain is then an integral domain formed by the adjunction of an algebraic integer to the ordinary integral domain.

In the ordinary integral domain the theorem that a number can be resolved into its positive prime factors in only one way is fundamental. That the above theorem is not valid in all domains is well known and may be easily shown by an example. For instance, $6=2\times 3=(1+\sqrt{-5})\,(1-\sqrt{-5})$, the factors of each product being primes in the domain considered. It was in order to avoid such possibilities in factoring, as indicated by the above example, that the theory of ideals was created.

It is also generally known that numbers prime in one domain may be composite in another. For example, in the ordinary integral complex domain every rational prime of the form $4n+3$ is a complex prime and every rational prime of the form $4n+1$ is composite.

In the *Nouvelles Annales de Mathematiques* for 1903, M. G. Fontené

has considered algebraic integers of the form $x+y\sqrt{-5}$. In this article it is shown that factoring in the domain of the numbers $x+y\sqrt{-5}$ is not unique, but if the domain is enlarged so as to contain numbers of the form $\frac{x+y\sqrt{-5}}{\sqrt{2}}$ factoring becomes a unique process.

The relation which exists between factoring in a quadratic complex domain and the number of classes of quadratic forms corresponding to the domain has been considered by A. E. Westun in "Certain Systems of Quadratic Complex Numbers," *Transactions of the Cambridge Philosophical Society,* Vol. XVII, 1899. In both of the articles mentioned, factoring is not restricted to operations in an integral quadratic domain.

From the previous definitions and illustrations which are quite generally known, it is evident that an integral algebraic domain can be found by the adjunction of $\sqrt{-6}$ to the ordinary integral domain. This integral algebraic domain will be denoted for brevity by the letter Ω, and some of the properties of numbers in the domain will be considered with special reference to factoring.

We now prove the following theorem:

THEOREM I. *All ordinary primes of the form* $24z+1$ *and* $24z+7$ *are composite in* Ω.

From the theory for binary quadratic forms, it is known that D, the determinant of the form, must be a quadratic residue of m, where m is the number to be represented by the form. The number -6 is a quadratic residue of primes of the form $24z+1$, $24z+7$, $24z+5$, and $24z+11$. There are only two reduced forms for $D=-6$, namely, x^2+6y^2 and $2x^2+3y^2$. The form $2x^2+3y^2$ will not give complex factors in Ω, and so need not be considered. Then all ordinary primes which are composite in Ω must be represented by x^2+6y^2.

If m is the prime to be represented, $m=x^2+6y^2$ and, therefore, $x^2\equiv m \pmod{6}$. $x^2\equiv 1 \pmod{6}$, but x^2 is not congruent to 5 (mod. 6) and, therefore, all ordinary primes represented by x^2+6y^2 are of form $24z+1$ and $24z+7$ and only primes of these forms can be factored in Ω.

$$(1-\sqrt{-6})(1+\sqrt{-6})=7; \quad (5-\sqrt{-6})(5+\sqrt{-6})=31.$$

THEOREM II. *Primes of the form* $24z+1$ and $24z+7$ *can be resolved into their complex prime factors in only one way in* Ω.

There are four representations of x^2+6y^2 which give the same prime m. There are two solutions of the congruence $n^2\equiv -6 \pmod{m}$, and two substitutions transforming x^2+6y^2 into an equivalent form. Therefore, a prime number can be represented by x^2+6y^2 in only one way and consequently primes of the form $24z+1$ and $24z+7$ can be resolved into prime complex factors in only one way in Ω.

THEOREM III. *In order that a composite rational integer may be resolved into complex factors only in Ω, it must be of the form, $m = a^{(\alpha)} b^{(\beta)} c^{\gamma} d^{\delta}$, where a represents primes of the form $24z+1$ and $24z+7$, and (α) the number of primes in a; b represents primes of the form $24z+5$ and $24z+11$ and (β) the number of primes in b; $c=2$, $d=3$, $(\beta)+\gamma+\delta$ is an even number and $\gamma+\delta \not> (\beta)$.*

The determinant of this form, that is, -6, must be a quadratic residue of any composite number m properly represented by x^2+6y^2 and, therefore, must be a quadratic residue of every prime factor of m. $x^2+6y^2=m$, or m must be a quadratic residue of 6. The number -6 is a quadratic residue of primes of the form $24z+1$, $24z+7$, $24z+5$, and $24z+11$. Primes of form $24z+1$ and $24z+7$ are quadratic residues of 6, while primes of the form $24z+5$ and $24z+11$ are non-quadratic resides of 6. Then any number $m = a^{(\alpha)} b^{(\beta)}$ is properly representable by x^2+6y^2, or may be resolved into complex factors only in Ω, (β) being even, since an even number of non-quadratic residues is a quadratic residue. We have the following equations:

(I). $2x^2+3y^2=(x\sqrt{2}+y\sqrt{-3})(x\sqrt{2}-y\sqrt{-3})$.

(II). Then $2(2x^2+3y^2)=\sqrt{2}(x\sqrt{2}+y\sqrt{-3}).\sqrt{2}(x\sqrt{2}-y\sqrt{-3})$
$=(2x+y\sqrt{-6})(2x-y\sqrt{-6})=(z+y\sqrt{-6})(z-y\sqrt{-6})$.

(III). Also, $3(2x^2+3y^2)=\sqrt{3}(y\sqrt{3}+x\sqrt{-2}).\sqrt{3}(y\sqrt{3}-x\sqrt{-2})$
$=(3y+x\sqrt{-6})(3y-x\sqrt{-6})=(z'+x\sqrt{-6})(z'-x\sqrt{-6})$.

The primes 2 and 3 are not represented by x^2+6y^2 and are not properly represented by $2x^2+3y^2$. Equations (II) and (III) then show that neither 2^{γ}, 3^{δ}, nor $2^{\gamma}3^{\delta}$ are properly represented, or have complex factors in the domain Ω. Primes of the form $24z+5$ and $24z+11$, however, are properly represented by $2x^2+3y^2$ and an inspection of equations (II) and (III) will show that the product of any such prime b by either 2 or 3 will give a number having complex factors in Ω. Since neither 2^{γ}, 3^{δ} nor $2^{\gamma}3^{\delta}$ have complex factors in Ω, but $2b$ and $3b$ have such factors, $\gamma+\delta$ cannot be greater than (β), $(\beta)-\gamma+\delta$ must be even since all primes of form $24z+5$ and $24z+11$ are non-quadratic residues, (mod. 6). Therefore, $(\beta)+\gamma+\delta$ must be an even number.

THEOREM IV. *Any composite rational integer $m=a^{(\alpha)} b^{(\beta)} c^{\gamma} d^{\delta}$ representable by the form x^2+6y^2 can be resolved into its prime complex factors in more than one way provided it contains at least two different prime factors of the form* $24z+5$ and $24z+11$, *so that* $(\beta)-(\gamma+\delta) \not< \alpha$.

Let Kx represent the composite rational integer, x representing the product of prime factors of form $24z+5$ and $24z+11$, K the product of all other prime factors. K can be resolved into its complex prime factors in only one way because primes of form $24z+1$ and $24z+7$ are resolvable into

complex factors in only one way and 2^γ, 3^δ, or $2^\gamma 3^\delta$ have no complex factors in the domain. It is also evident from the previous theorems that no prime of the form $24z+1$ and $24z+7$ combined with either 2 or 3 or with a single prime of the form $24z+5$ or $24z+11$ can be resolved into complex factors only in Ω. $2b$ and $3b$, where b is any prime of form $24z+5$ and $24z+11$, can be resolved into complex factors in only one way since 2 and 3 have no complex factors in Ω. From these statements, it is clear that the factoring in different distinct ways must depend only on the factors of x.

A number x can be represented or resolved into its complex factors in Ω in 2^{w-1} ways where w represents the number of different prime factors of x. These $w-1$ representations will be distinct, since primes of form $24z+5$ and $24z+11$ are also primes in the domain Ω.

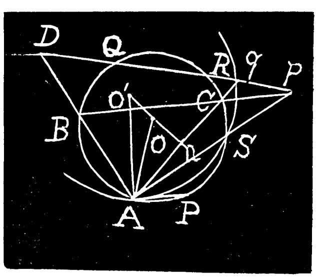
NOTE ON THE STEINER POINT.

By W. GALLATLY, Swanage, England.

Let ABC be the mid-point triangle, and PQR the pedal triangle of a given triangle $A'B'C'$, so that AP is parallel to BC, and $AQ=AR=BC=a$: $(a>b>c)$. And since rq is antiparallel to BC, $\angle Arq=C$, $\angle Aqr=B$.

Describe a circle around $\triangle Aqr$. This touches AP since $\angle Arq=C=\angle CAP$, and therefore the center O' lies on the perpendicular from A on BC. Join Ap, cutting the circle ABC in S. Then S is the Steiner point of the triangle ABC.

Since q, C, B, r are cyclic, $pq.pr=pC.pB=pS.pA$. Hence S lies on the circle Aqr. To find the radius, ρ, of this circle, we have $Aq=2\rho.\sin Arq$ $=2\rho.\sin C$. Also $Aq=AR.\dfrac{\sin A}{\sin B}=2R\sin A.\dfrac{\sin A}{\sin B}$, where R is the radius of the circle ABC. Therefore $\rho=R.\dfrac{a^2}{b^2}$.



To determine the length of AS.

In the triangle OAO', $OA=R$, $O'A=R.a^2/bc$, $\angle OAO'=\angle B-\angle C$, and $OO'^2=OA^2+O'A^2-2.OA.O'A.\cos\angle OAO'$.

But $\cos B\cos C=\dfrac{a^4-(b^2-c^2)^2}{4a^2bc}$, $\sin B\sin C$ $=\dfrac{16\triangle^2}{4a^2bc}$, where $\triangle^2$ is the square of the area of the triangle ABC.

Hence, $\dfrac{OO'^2}{R^2}.b^2c^2=a^4+\ldots-b^2c^2-\ldots=\dfrac{a^2b^2c^2.e^2}{4R^2\sin^2\omega}$, where e is the eccentricity of the Brocard ellipse, and ω is the Brocard angle.

Hence $OO'=\frac{1}{2}.ae.\text{cosec}\,\omega$. But $An.OO'=AO.AO'.\sin OAO'$.

Therefore $AS=2.An=\dfrac{2R\sin\omega}{e}.\dfrac{a(b^2-c^2)}{abc}$.

To determine the length of BS. Since AS is the radical axis of the two circles, we have $2.OO'$ times the perpendicular from B on AS equals the power of B for the circle Aqr, that is, equals $BA.Br$.

Hence $2.OO'.SB\sin C=c.BQ.\dfrac{\sin B}{\sin C}=b.R\sin(A-C)$. So that $SB=\dfrac{2R\sin\omega}{e}.\dfrac{b\ (a^2-c^2)}{abc}$. Similarly, $SB=\dfrac{2R\sin\omega}{e}.\dfrac{c\ (a^2-b^2)}{abc}$.

Hence the point S is unique, and its trilinear coordinates are

$$\frac{1}{a\ (b^2-c^2)},\ \frac{1}{b\ (c^2-a^2)},\ \frac{1}{c\ (a^2-b^2)}.$$

Therefore the circles described with radii $R.ac/b^2$ and $R.ab/c^2$ to touch BQ and CR, respectively in B and C, will also pass through S.

The feet of the perpendiculars from S on the sides of the triangles ABC and PQR are collinear. Let these feet be represented by x, y, z, x', y', z'. The diagram shows that y, z, and x' are on the Simson line of S for the triangle Aqr. Drawing the second circle, touching BQ in B, we see that z, x, and y' are collinear, and likewise for x, y, and z'.

Hence the six points are collinear.

The distances SP, SQ, SR may be readily found. Let $\angle SAP=\theta$.

Then $\sin\theta=\cos O'An=\dfrac{An}{O'A}=\dfrac{R\sin\omega}{e}.\dfrac{a\ (b^2-c^2)}{abc}.\dfrac{1}{R}.\dfrac{b\ c}{a^2}=\dfrac{\sin\omega}{e}.\dfrac{b^2-c^2}{a^2}$.

Hence $SP=2RS\sin\theta=\dfrac{2R\sin\omega}{e}.\dfrac{b^2-c^2}{a^2}$.

If p be the perpendicular from S on the Simson line of S for ABC, and if SA, SB, SC make angles α, β, γ with the diameter through S, it is known that

$$p=2\,R.\cos\alpha.\cos\beta.\cos\gamma$$

$$=2R.\frac{SA}{2\,R}.\frac{SB}{2\,R}.\frac{SC}{2\,R}=2R.\left(\frac{\sin\omega}{e}\right)^3.\frac{(b^2-c^2)\,(a^2-c^2)\ (a^2-b^2)}{a^2b^2c^2}.$$

And for the triangle PQR, $p'=2R.\left(\dfrac{\sin\omega}{e}\right)^3.\dfrac{(b^2-c^2)\,(a^2-c^2)\ (a^2-b^2)}{a^2b^2c^2}$.

Therefore, $p=p'$, as already demonstrated.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

300. **Proposed by J. F. LAWRENCE, A. B., Professor of Mathematics, Stillwater, Okla.**

If α, β, γ, ..., are the roots of the equation $\sin mx - nx\cos mx = 0$, prove that $\tan^{-1}\frac{x}{\alpha}+\tan^{-1}\frac{x}{\beta}+\ldots+\tan^{-1}\frac{x}{\nu}=0$.

Solution by B. F. FINKEL, Ph. D., Drury College, Springfield, Mo.

This is problem 16, p. 328, Loney's *Trigonometry*. The problem is not clearly stated. In a letter December 2, 1908, from the author, he says he cannot now recall what he had in mind when he proposed the problem. As the problem reads, one would infer that if any ν roots of the given equation be taken and arcs formed whose tangents are any number, x, divided by these roots, the sum of these arcs is zero. But this is manifestly incorrect.

There is one way in which the statement is correct, viz: Suppose $\sin mx$ and $\cos mx$ be developed in series. The given equation then becomes

$$(m-n)x+m^2\left(\frac{n}{2\,!}-\frac{m}{3\,!}\right)x^3-m^4\left(\frac{n}{4\,!}-\frac{m}{5\,!}\right)x^5+\ldots\ ad\ infinitum=0\ldots(1).$$

One root of this equation is 0. Dividing the equation by x, we obtain

$$(m-n)+m^2\left(\frac{n}{2\,!}-\frac{m}{3\,!}\right)x^2-m^4\left(\frac{n}{4\,!}-\frac{m}{5\,!}\right)x^4+\ldots=0\ldots(2).$$

The roots of this equation, of which there are an infinite number, enter in pairs with opposite signs. Thus if $\alpha, \beta, \gamma, \ldots$ are roots, so are $-\alpha$, $-\beta$, $-\gamma$, ..., since the left hand member is an even function of x. Then, if the root, 0, is excluded, and if no arc exceeds, in absolute value, π radians, we have

$$\left.\tan^{-1}\frac{x}{\alpha}+\tan^{-1}\frac{x}{-\alpha}+\tan^{-1}\frac{x}{\beta}+\tan^{-1}\frac{x}{-\beta}+\ldots+\tan^{-1}\frac{x}{\nu}+\tan^{-1}\frac{x}{-\nu}\right]_{\nu=\infty}=0.$$

The statement is also true if we take any $\nu/2$ pairs of roots.

If ν is increased indefinitely so that all roots except 0 are included, the statement, that the sum of the arcs$=n'\pi$, may be shown to be true, as follows: We have

$$\tan\left[\tan^{-1}\frac{x}{\alpha}+\tan^{-1}\frac{x}{\beta}+\tan^{-1}\frac{x}{\gamma}+\ldots+\tan^{-1}\frac{x}{\nu}\right].$$

$$=\frac{x\ \Sigma\frac{1}{\alpha}-x^3\ \Sigma\frac{1}{\alpha\beta\gamma}+x^5\ \Sigma\frac{1}{\alpha\beta\gamma\delta\varepsilon}-\ldots}{1-x^2\ \Sigma\frac{1}{\alpha\beta}+x^4\ \Sigma\frac{1}{\alpha\beta\gamma\delta}-\ldots}=0\ldots(3);\text{ for}$$

$$\Sigma\frac{1}{\alpha}\equiv\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}+\ldots=\frac{\beta\gamma\delta\varepsilon\ldots+\alpha\gamma\delta\epsilon\ldots+\alpha\beta\delta\epsilon\ldots+}{\alpha\beta\gamma\delta\varepsilon\ldots}.$$

The numerator of this fraction is the sum of the roots taken one less than all at a time, and is therefore equal to the coefficient of x which in equation (2) is zero, and the denominator is the product of the roots which in (2) is the known term, $m-n$. Hence, $\Sigma\frac{1}{\alpha}=0$. Similarly, $\Sigma\frac{1}{\alpha\beta\gamma}$ is a fraction whose numerator is the sum of the products of the roots taken three less than all at a time, and is therefore the coefficient of x^3 in (2), which is 0. Hence, $\Sigma\frac{1}{\alpha\beta\gamma}=0$. Similarly, for the other terms of the numerator of (3). From similar considerations, $\Sigma\frac{1}{\alpha\beta}$, $\Sigma\frac{1}{\alpha\beta\gamma\delta}$, $\ldots\neq0$.

Hence, $\tan\Sigma\left(\tan^{-1}\frac{x}{\alpha}\right)=0$. Hence, $\Sigma\tan^{-1}\frac{x}{\alpha}=n'\pi$ where n' is any integral positive or negative number.

Also solved by G. B. M. Zerr, and V. M. Spunar.

301. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

A is at Philadelphia, B at Chicago. A's personal equation is e; B's is E. When a star crosses A's meridian at time $t_1=8$ hours, 33 minutes, 24 seconds, he presses a button, telegraphing the fact to B, who receives it at time $t_2=7$ hours, 43 minutes, 23 seconds. When it crosses B's meridian at time $T_2=8$ hours, 33 minutes, 10 seconds, he telegraphs A, who receives it at time $T_1=9$ hours, 23 minutes, 11 seconds. They now exchange places, and on the second day following, B observes the transit at time $t'_1=8$ hours, 33 minutes, 26 seconds, and A gets the information at Chicago at time $t'_2=$ 7 hours, 43 minutes, 25 seconds. It crosses A's meridian at time $T'_2=8$ hours, 33 minutes, 12 seconds, and B gets the information at time $T'_1=9$ hours, 23 minutes, 13 seconds. Find the difference of longitude between Philadelphia and Chicago.

Solution by the PROPOSER.

The true times of the two transits at Philadelphia are t_1+e and T_1.

Hence, difference of time between Philadelphia and Chicago is $D=T_1+\ldots-t_1-e\ldots(1)$.

The true times of the two transits at Chicago are T_2+E and t_2.

Hence. $D=T_2+E-t_2\ldots(2)$.

$(1)+(2)$ gives $2D=T_1+T_2-t_1-t_2+E-e\ldots(3)$.

Similarly, when A and B have changed places, $2D=T'_1+T'_2-t'_1-t'_2-E+e\ldots(4)$.

$\frac{1}{4}(3)+(4)$ gives $D=\frac{1}{4}(T_1+T_2+T'_1+T'_2-t_1-t_2-t'_1-t'_2)$. Substituting the values of the T's and t's, we have

$D=\frac{1}{4}$(3 hours, 19 minutes, 8 seconds) $=49$ minutes, 47 seconds, and $L=15D=12^\circ\ 26'\ 45''$, the difference of longitude.

Also solved by V. M. Spunar.

GEOMETRY.

335. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Determine analytically, the point where three lines in a plane appear of equal length.

Solution by the PROPOSER.

In the most general solution we can take the rectangular axes of coordinates so that the axis of abscissas agrees, in direction, with one of the lines. Let this line be given by (0, 0); $(a, 0)$; the second and third lines are then represented by the coordinates (b, c); (d, f), and (h, k); (m, n), respectively. Also let (x, y) be the point where they appear equal. The tangent of the angle formed by the lines through (x, y); (b, c) and (x, y); (d, f) is

$$\frac{\dfrac{c-y}{b-x}-\dfrac{f-y}{d-x}}{1+\dfrac{(c-y)\ (f-y)}{(b-x)\ (d-x)}}=\frac{cd-bf+(f-c)x+(b-d)y}{bd+cf-(b+d)x-(c+f)y+x^2+y^2}.$$

With similar expressions for the other lines.

$$\therefore \frac{ay}{ax-x^2-y^2}=\frac{cd-bf+(f-c)x+(b-d)y}{bd+cf-(b+d)x-(c+f)y+x^2+y^2}\ldots(1).$$

$$\frac{ay}{ax-x^2-y^2}=\frac{km-hn+(n-k)x+(h-m)y}{hm+kn-(h+m)x-(k+n)y+x^2+y^2}\ldots(2).$$

$$\begin{aligned}\therefore (f-c)x^3&+(cd-bf+ac-af+by-dy+ay)x^2\\&+(abf-acd-2aby+fy^2-cy^2)x+(abd+acf)y\\&+(cd-bf-ac-af)y^2+(a+b-d)y^3=0\ldots(3).\end{aligned}$$

$$(n-k)x^3+(km-hn+ak-an+hy-my+ay)x^2$$
$$+(ahn-akm-2ahy+ny^2-ky^2)x+(ahm+akn)y$$
$$+(km-hn-ak-an)y^2+(a+h-m)y^3=0\ldots(4).$$

(3) and (4) are the equations to determine x, y.

I. Let $c=f=k=n=0$. Then the three lines lie in the same straight line.

(3) becomes $(a+b-d)(x^2+y^2)=2abx-abd\ldots(5)$.

(4) becomes $(a+h-m)(x^2+y^2)=2ahx-ahm\ldots(6)$.

$$\therefore x=\frac{m(ah+bh+bd)-d(hm+ab+bd)}{2(ah-ab-dh+bm)}.$$

$$y=\pm\sqrt{\frac{2a(b-h)x+a(hm-bd)-(b-d-h+m)x^2}{b-d-h+m}}.$$

Let $a=b$, $h=$d. Then the three lines form one consecutive straight line.

$$\therefore x=\frac{3adm-d^2m-a^2d-ad^2}{2(ad-a^2-d^2+am)}$$

$$y=\pm\sqrt{\frac{2a(a-d)x+a(dm-ad)-(a-2d+m)x^2}{a-2d+m}}.$$

Let $d=2a$, $m=3a$. Then the lines are all of equal length.

$\therefore x=0/0$, and is indeterminate; $y=\infty$.

Hence there is no point at which three equal straight lines forming a continuous straight line will appear equal to each other.

II. $b=a$ and $c=0$ in (3) gives

$$(fx-2af+2ay-dy)(x^2+y^2)+a^2(fx-2xy+dy)=0\ldots(7).$$

$h=d$, $k=f$, $m=n=0$ in (4) gives

$$(fx-af-ay-dy)(x^2+y^2)+2ay(dx+fy)=0\ldots(8).$$

The three lines now are the sides of a triangle. The solution of these equations in general terms would be quite difficult, yet one can determine the values of x and y by the following method: Let u, v, w be the distances from (x, y) to $(0, 0)$; $(a, 0)$, (d, f), respectively. Then we have

$$x^2+y^2=u^2,\quad (a-x)^2+y^2=v^2,\quad (d-x)^2+(f-y)^2=w^2.$$

$$\therefore\ x=\frac{a^2+u^2-v^2}{2a},\quad y=\frac{(u^2-ad)(a-d)+dv^2+af^2-aw^2}{2af}.$$

Also, $u^2+uv+v^2=a^2$,
$u^2+uw+w^2=d^2+f^2$,
$v^2+vw+w^2=(a-d)^2+f^2$.

$$\therefore\ u=\frac{ad\pm\frac{1}{3}af\sqrt{3}}{\sqrt{(a^2+d^2+f^2-ad\pm af\sqrt{3})}},$$

$$v=\frac{a^2-ad\pm\frac{1}{3}af\sqrt{3}}{\sqrt{(a^2+d^2+f^2-ad\pm af\sqrt{3})}},$$

The plus sign is applicable to our solution.

$$w=\frac{d^2+f^2-ad\pm\frac{1}{3}af\sqrt{3}}{\sqrt{(a^2+d^2+f^2-ad\pm af\sqrt{3})}},$$

Let $d=\frac{1}{2}a$, and $f=\frac{1}{2}a\sqrt{3}$. Hence, $u=\frac{1}{3}a\sqrt{3}=v=w$. Thus the values of x and y are determined, and equations (7) and (8) are solved.

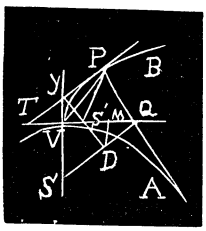
Hence $x=\dfrac{3ad^2+3af^2+3a^2d+a^2f\sqrt{3}+4adf\sqrt{3}}{6(a^2+d^2+f^2-ad+af\sqrt{3})}=\frac{1}{2}a$, when $d=\frac{1}{2}a$ and $f=\frac{1}{2}a\sqrt{3}$.

$y=\dfrac{3a^2d\sqrt{3}-3ad^2\sqrt{3}+af^2\sqrt{3}+3a^2f}{6(a^2+d^2+f^2-ad+af\sqrt{3})}=\frac{1}{6}a\sqrt{3}$, when $d=\frac{1}{2}a$, $f=\frac{1}{2}a\sqrt{3}$.

Also solved by C. N. Schmall.

323. Proposed by W. J. GREENSTREET, Marling School, Stroud, England.

S, S' are the foci of two co-vertical parabolas A and B, the axes of which are at right angles. Draw the circle K on SS' as diameter. K is cut in D and E by a straight line parallel to the axis of A such that S' lies midway between it and that axis. Show that the lines $S'D$, $S'E$ are parallel to the two tangents to A which are normals to B.



II. Solution by (1) PROPOSER, and (2) R. F. DAVIS, M. A.

I°. The axis of B is the tangent at the common vertex V to A. Hence if PQ be normal at P to B, meeting the axis of B in Q, and if PQ is at the same time a tangent to A, it necessarily follows that SQ is perpendicular to PQ and therefore parallel to TP (the tangent at P to B). Or,

II°. $S'T=S'P=S'Q$ (letters as above). Draw $YS'D$ perpendicular to the parallels TP, SQ, and DM perpendicular to the axis of B (Y, of course, being on the tangent at V to B). Then $S'M=VS'$, and D is on the circle, SS' diameter, since $\angle SDS'=90°$.

III. **Solution by the PROPOSER.**

The foci S, S' of (A), $y^2=4ax$, and (B) $x^2=4by$, are, respectively, $(a, 0)$, $(0, b)$, and the circle in question has as its equation $x^2+y^2-ax-by=0$.

Since the common vertex V is $(0, 0)$ and S' is $(0, b)$, the point M in which DE cuts the axis of B is $0, 2b$, and the line DE is $y=2b$.

This cuts the circle where $x^2-ax+2b^2=0$...(1), giving D and E. If the roots of this be x_1 and x_2, D, E are $(x_1, 2b)$, $(x_2, 2b)$, and $S'D$, $S'E$ have respectively, for their equations,

$$bx-x_1y+x_1b=0 \text{ and } bx-x_2y+x_2b=0.$$

The tangent at $(at^2, 2at)$ to A is $x-ty+at^2=0$.
The normal at $(2bt', at'^2)$ to B is $x+t'y-2bt'-bt'^3=0$.
These are the same line if $1=\dfrac{-t}{t'}=\dfrac{-at^2}{bt'(2+t'^2)}$.
$\therefore t=-t'$, and rejecting the values $t=t'=0$ we have $t^2b-at+2b=0$... (2).
The roots of (1) and (2) are the same, and the property is proved.
Perhaps Prof. Zerr will reconsider the solution he offers.

CALCULUS.

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263. **Proposed by V. M. SPUNAR, M. S., C. E., East Pittsburg, Pa.**

Find a point such that the sum of the squares of its distances from n given points shall be a minimum, and prove that the value so found is $1/n$th part of the sum of the squares of the mutual distances between the n points, taken two and two.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

This problem is proposed and solved in Williamson's *Differential Calculus*, 9th edition, page 192, Art. 157.

Taking one of the points as origin, and the axes rectangular, let (x, y) be the coordinates of the required point. Let (a_1, b_1); (a_2, b_2); (a_3, b_3); ..., (a_{n-1}, b_{n-1}) be the coordinates of the other $(n-1)$ points. Then

$$x^2+y^2+(x-a_1)^2+(y-b_1)^2+(x-a_2)^2+(y-b_2)^2+\ldots$$
$$+(x-a_{n-1})^2+(y-b_{n-1})^2=u=\text{minimum, or}$$

$$nx^2+ny^2-2(a_1+a_2+\ldots+a_{n-1})x-2(b_1+b_2+\ldots+b_{n-1})y$$
$$+a_1^2+a_2^2+\ldots+a^2_{n-1}+b_1^2+b_2^2+\ldots+b^2_{n-1}=u=\text{minimum.}$$

$$\therefore (nx-a_1-a_2-\ldots-a_{n-1})dx+(ny-b_1-b_2-\ldots-b_{n-1})dy=0.$$

$$\therefore x=\frac{a_1+a_2+\ldots+a_{n-1}}{n},\quad y=\frac{b_1+b_2+\ldots+b_{n-1}}{n}.$$

The point is, therefore, the center of mean position of the n points as stated.

Also solved by J. Scheffer.

264. Proposed by **W. J. GREENSTREET, M. A., Stroud, England.**

The join of the center of curvature of a curve to the origin is at α to the initial line. Prove that with the usual notation:

$$\frac{d\alpha}{d\psi}\left[\left(\frac{dp}{d\psi}\right)^2+\left(\frac{d^2p}{d\psi^2}\right)^2\right]=\frac{dp}{d\psi}.\frac{d\rho}{d\psi}.$$

No solution of this problem has been received.

265. Proposed by **V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.**

Find two curves which possess the property that the tangents TP and TQ to the inner one always make equal angles with the tangent TT' to the outer.

No solution of this problem has been received.

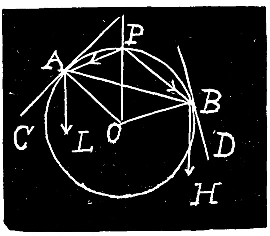
MECHANICS.

219. Proposed by **W. J. GREENSTREET, M. A., Stroud, England.**

A rod length $a\sqrt{3}$, weight W, has at each end a smooth ring which can slide on a vertical circle radius r. Each ring is attached by an elastic string (natural lengths a, b; moduli μa, μb) to the highest point of the circle. Find the inclination of the rod to the horizon in a position of equilibrium.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

In what follows we regard the strings as having no weight, and also that both strings are in tension from the weight of the rod and rings, and the rod is above the center of the circle. Let $AB=a\sqrt{3}=$rod; $AP=$string a; $BP=$string b; O, the center of the circle, radius $AO=r$. Draw AK perpendicular to PO. Let $m=$ weight of each ring; $\angle AOB=\beta=2\sin^{-1}(a\sqrt{3}/2r)$; $\angle APB=\pi-\frac{1}{2}\beta$; $\angle KAB=\theta=$angle AB makes with the horizon; $\angle PAB=\phi$; $\angle PBA=\frac{1}{2}\beta-\phi$; $T=$tension of string AP; $T'=$tension of string PB. When in equilibrium, W and the components of T, T' tangent at A, B meet in a point. $AP=a(1+T/\mu a)$ $=(\mu a+T)/\mu$, $BP=(\mu b+T')/\mu$, $T=(\frac{1}{2}W+m)\sin(\phi-\theta)$. Let $(\frac{1}{2}W+m)=Q$.



$\therefore T=Q\sin(\phi-\theta)$, $T'=Q\sin(\theta+\frac{1}{2}\beta-\phi)$, $AP/PB=(\mu a+T)/(\mu b+T')=\sin(\frac{1}{2}\beta-\phi)/\sin\phi$... (1).

$3\mu^2a^2=(\mu a+T)^2+(\mu b+T')^2+2(\mu a+T)(\mu b+T')\cos\frac{1}{2}\beta$... (2).

The values of T and T' in (1) and (2) give

$$[\mu a+Q\sin(\phi-\theta)]\sin\phi=[\mu b+Q\sin(\theta+\tfrac{1}{2}\beta-\phi)]\sin(\tfrac{1}{2}\beta-\phi)\ldots(3).$$

$$3\mu^2a^2=[\mu a+Q\sin(\phi-\theta)]^2+[\mu b+Q\sin(\theta+\tfrac{1}{2}\beta-\phi)]^2$$
$$+2[\mu a+Q\sin(\phi-\theta)][\mu b+Q\sin(\theta+\tfrac{1}{2}\beta-\phi)]\cos\tfrac{1}{2}\beta \ldots (4).$$

Eliminating ϕ between (3) and (4) gives an equation to determine θ.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

152. Proposed by H. S. VANDIVER, Bala, Pa.

When p is a prime of the form $5n\pm1$ then there is a positive integer a such that $a^2\equiv5(\text{mod } p)$. Show that $\left(\frac{a\pm1}{p}\right)=\pm\left(\frac{-2a}{p}\right)$, according as p is of the form $5n+1$ or $5n-1$.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A general expression involving all cases is, apparently, not easily deduced. The following cases hold.

(1) Let $n=2$. Then $p=5n+1=11$. $a=4$, $a=p-4=7$, $a=2p-7=15$, $a=3p-15=18$, etc.

(2) Let $n=4$. Then $p=5n-1=19$, $a=9$, $a=p-9=10$, $a=2p-10=28$, $a=3p-28=29$, etc.

(3) Let $n=6$. Then $p=5n+1=31$, $a=6$, $a=p-6=25$, $a=2p-25=37$, $a=3p-37=56$, etc.

(4) Let $n=8$. Then $p=5n+1=41$, $a=13$, $a=p-13=28$, $a=2p-28=54$, $a=3p-54=69$, etc.

(5) Let $n=12$. Then (b) $p=5n+1=61$; (c) $p=5n-1=59$.

(b) $a=26$, $a=p-26=35$, $a=2p-35=87$, $a=3p-87=96$, etc.

(c) $a=8$, $a=p-8=51$, $a=2p-51=67$, $a=3p-67=110$, etc.

For every value of n that makes $5n\pm1$ a prime, we can find values for a satisfying the condition. It is also easy to see that a can have an infinitude of values for each case.

In (1), (3), (4), (5)(b), $(a+1)^{\frac{1}{2}(p-1)}=(-2a)^{\frac{1}{2}(p-1)}\equiv-1(\text{mod. } p)$.

In (2), (5)(c), $(a-1)^{\frac{1}{2}(p-1)}=-(-2a)^{\frac{1}{2}(p-1)}\equiv1(\text{mod. } p)$.

$\left(\frac{a\pm1}{p}\right)=\pm\left(\frac{-2a}{p}\right)$, in the cases examined, according as $p=5n\pm1$.

A general solution is desired. ED. F.

152. Proposed by H. S. VANDIVER, Bala, Pa.

Prove geometrically:

$$\sum_{n=1}^{\frac{1}{2}(p-1)}\left[\frac{n^2}{p}\right]=\frac{p-3}{4}\cdot\frac{p-1}{2}-\sum_{n=1}^{\frac{1}{4}(p-4}\left[\sqrt{np}\,\right],$$ where $p\equiv3(\text{mod. } 4)$ and $\left[\frac{k}{p}\right]$ represents the greatest integer in k/p.

Remarks by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

In the second term of the second member of the equation, $\frac{1}{4}(p-4)$ should be $\frac{1}{4}(p-3)$. The equation is generally but not universally true as is shown by induction in what follows.

Let $p=4m+3$, then $\frac{1}{2}(p-1)=2m+1$, $\frac{1}{4}(p-3)=m$,

$$\sum_{n=1}^{\frac{1}{2}(p-1)}\left[\frac{n^2}{p}\right]=\frac{p-3}{4}\cdot\frac{p-1}{2}-\sum_{n=1}^{\frac{1}{4}(p-3)}[\sqrt{(np)},$$

or as follows:

$$\begin{array}{rr} & A=\ B-C, \\ m=2, & 3=\ 10-7, \\ m=3, & 7=\ 21-14, \\ m=4, & 11=\ 36-25, \\ m=5, & 18=\ 55-37, \\ m=7, & 34=105-71. \end{array}$$

If one of the $[n^2/p]$ is an exact quotient, and hence one of the $[\sqrt{(np)}]$ rational, the equation is $A=1+B-C$.

$$m=6,\ p=27,\ [9^2/p]=3,\ \sqrt{(3\times27)}=9,$$
$$m=15,\ p=63,\ 21^2/p=7,\ \sqrt{(7\times63)}=21.$$

$\therefore A=1+B-C$, $m=6\ldots25=1+78-54$, $m=15\ldots153=1+465-313$.

If two of the $[n^2/p]$ are exact quotients, and hence two of the $[\sqrt{(np)}]$ rational, the equation becomes $A=2+B-C$.

$$m=18,\ p=75,\ 15^2/p=3,\ \sqrt{(3\times p)}=15,\ 30^2/p=12,\ \sqrt{(12\times p)}=30.$$

$\therefore A=2+B-C$ becomes $219=2+666-449$ for $m=18$. $A=t+B-C$ is the true universal equation.

The geometric proof in this solution is wanting. Who can produce it? ED. F.

155. Proposed by PROF. R. D. CARMICHAEL, Anniston, Alabama.

If p and q are primes and m and n are any integers, find the cases in which the equation $p^m-q^n=1$ may be satisfied.

Remarks by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Some values, found by inspection, are given in the following table:

p	q	m	n
3	2	1	1
3	2	2	3
2	1	1	1
2	31	5	1
2	127	7	1

ERRATUM. In 156 for e^2 read e^3.

AVERAGE AND PROBABILITY.

196. Proposed by R. D. CARMICHAEL, Anniston, Ala.

A circle is inscribed in a square. Find the chance that the distance between two points within the square and without the circle shall not exceed a side of the square.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let $2a$=side of square; (x, y), (u, v) the coordinates of the points; $\sqrt{(a^2-x^2)}=y_1$, $\sqrt{(a^2-u^2)}=v_1$. If both points are in the same corner, the distance between them is always less than $2a$. If both points are situated one each in opposite corners, the distance between them is always greater than $2a$. If both points are placed one each in adjacent corners, we have $\sqrt{[(x-u)^2+(y+v)^2]}=4a^2$, for the greatest distance between the points and $\sqrt{\{(x-u)^2+[\sqrt{(a^2-x^2)}+\sqrt{(a^2-u^2)}]^2\}}=\sqrt{[(x-u)^2+(y+v)^2]}$, for the least distance between the points.

$\therefore v_2=\sqrt{[4a^2-(x-u)^2]}-y$, $v_3=\sqrt{(a^2-x^2)}+\sqrt{(a^2-u^2)}-y$.

$$\therefore p=\frac{\int_0^a\int_0^a\left[\int_{y_1}^a\int_{v_1}^a dydv+\int_{y_1}^a\int_{v_1}^{v_2}dy\,dv\right]dxdu+\int_0^a\int_0^x\int_{y_1}^a\int_{v_3}^{v_2}dxdudydv}{3\int_0^a\int_0^a\int_{y_1}^a\int_{v_1}^a dx\,dy\,du\,dv}$$

$$=\frac{a^4(4-\pi)^2+0+.16\int_0^a\int_0^x\int_{y_1}^a\int_{v_3}^{v_2}dx\,du\,dy\,dv}{3a^4(4-\pi)^2}=\tfrac{1}{3}+\frac{1.6}{3a^4(4-\pi)^2}.M.$$

$$M=\int_0^a\int_0^x\int_{y_1}^a\{\sqrt{[4a^2-(x-u)^2]}-\sqrt{[a^2-x^2]}-\sqrt{[a^2-u^2]}\}dx\,du\,dy$$

$$=\int_0^a\int_0^x[a-\sqrt{(a^2-x^2)}]\{\sqrt{[4a^2-(x-u)^2]}-\sqrt{[a^2-x^2]}-\sqrt{[a^2-u^2]}\}dx\,du$$

$$=\int_0^a [a-\sqrt{(a^2-x^2)}]\left(\frac{x}{2}\sqrt{[4a^2-x^2]}-\tfrac{3}{2}x\sqrt{[a^2-x^2]}+2a^2\sin^{-1}\frac{x}{2a}\right.$$

$$\left.-\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right)dx=a^4\int_0^{\frac{1}{2}\pi}(1-\cos\theta)\,[\tfrac{1}{2}\sin\theta\sqrt{(4-\sin^2\theta)}-\tfrac{3}{2}\sin\theta\cos\theta$$

$$+2\sin^{-1}(\tfrac{1}{2}\sin\theta)-\tfrac{1}{2}\theta]\cos\theta\, d\theta=a^4(\tfrac{3}{2}\sqrt{3}-\tfrac{29}{12}+\frac{\pi^2}{32}+\frac{\pi}{12})$$

$$-\ a^4\int_0^{\frac{1}{2}\pi}[\tfrac{1}{2}\sin\theta\sqrt{)4-\sin^2\theta)}+2\sin^{-1}(\tfrac{1}{2}\sin\theta)]\cos^2\theta\, d\theta$$

$$=a^4(\tfrac{3}{2}\sqrt{3}-\tfrac{29}{12}+\frac{\pi^2}{32}+\frac{\pi}{12})-a^4\int_0^{\frac{1}{2}\pi}(2\sin\theta-\tfrac{1}{8}\sin^2\theta+\frac{15}{2^7}\sin^3\theta-\frac{1}{2^{10}}\sin^4\theta$$

$$+\frac{763}{2^{15}}\sin^5\theta-\frac{7}{2^{18}}\sin^6\theta+\frac{10219}{2^{22}}\sin^7\theta-\ldots)\cos^2\theta\, d\theta.$$

$$\therefore M=a^4(\tfrac{3}{2}\sqrt{3}-\tfrac{29}{12}+\frac{\pi^2}{32}+\frac{\pi}{12}-0.6595495)=0.0412517a^4.$$

$$\frac{16M}{3a^4(4-\pi)^2}=\frac{0.6600272}{2.2108850}=\tfrac{3}{10},\text{ nearly, }=.299.\quad \therefore p=\tfrac{1}{3}+\tfrac{3}{10}=\tfrac{19}{30},\text{ nearly.}$$

PROBLEMS FOR SOLUTION.

ALGEBRA.

208. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Find the conditions that the roots of $x^2+px+q=0$ may not lie between -1 and $+1$.

209. Proposed by PROFESSOR E. B. ESCOTT, Ann Arbor, Mich.

Solve, $bx^2+cy^2+az^2=ba^2+cb^2+ac^2$,
$cx^2+ay^2+bz^2=ab^2+bc^2+ca^2$,
$xyz=abc$.

210. Proposed by B. F. FINKEL, Ph. D., Drury College, Springfield, Mo.

Simplify, $\log[\sqrt[3]{(137)}\ \sqrt[7]{(56)}\div\sqrt[10]{(187)}\ \sqrt[4]{(75)}]$.

GEOMETRY.

341. Proposed by FRANK LOXLEY GRIFFIN, S. M., Ph. D., Instructor in Mathematics, Williams College.

Given $\rho=\cos(m/n)\theta$, where m and n are integers without a common factor. Deduce rules for finding by inspection:

(1) The angle between the beginning and end of any loop of this curve; (2) The number of distinct loops. [A loop is a portion of the curve between consecutive zero radii vectores.]

CALCULUS.

267. Proposed by FRANK LOXLEY GRIFFIN, S. M., Ph. D., Instructor in Mathematics, Williams College.

A point within an ellipse, upon a normal making an angle λ with the major axis, is arbitrarily chosen. With this point as pole, and the line through it parallel to the major axis as polar axis, the equation of the ellipse is, $A\cos^4\theta+B\cos^3\theta+C\cos^2\theta+D\cos\theta+E=0$, where the coefficients are functions of λ, of the radius vector ρ, and of the distance along the normal to the pole, ρ_1. Evidently for $\rho=\rho_1$, a solution is $\cos\theta=\cos\lambda$. Required the multiplicity of this solution for any values of ρ_1, $[\lambda\neq0,\ \rho_1\neq0]$.

268. Proopsed by PROFESSOR R. D. CARMICHAEL, Anniston, Ala.

Determine $\phi^{(y)}$, independent of u, so that the equation $\int_0^u(u-y)^{(2p-1)/2}\phi^{(y)}dy=u^m$ is satisfied, p and m being positive integers and $m>p$. Do you notice properties of special interest for any special cases?

MECHANICS.

223. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A sphere, radius $r=\frac{1}{3}$ inches, density $\delta=11.38$, falls from a height $h=500$ feet, into a lake depth $l=40$ feet. Find time of falling to surface of lake, time of falling from surface of lake to bottom, and total time of falling. Also the velocity at the bottom.

224. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A steel clock spring $w=\frac{7}{8}$ inch wide, $t=\frac{1}{32}$ inch thick, is wound around an axle $d=\frac{1}{4}$ inch in diameter. Find the greatest available moment for running the clock, using a factor of safety $f=6$.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

159. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

Show that if the equation $y^3=2x^2-1$ be possible in integers, $y=24n^2-1$, or $2n^2-1$, and find three solutions.

AVERAGE AND PROBABILITY.

200. Proposed by PROFESSOR R. D. CARMICHAEL, Anniston, Ala.

A line $AB=l$ is extended to P making $BP=p$. If a point D is taken at random in BP, what is the mean value of $AD.DP$?

201. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A random straight line is drawn across a circle and another through a given point on the circumference. Find the chance that they intersect within the circle.

202. Proposed by F. P. MATZ, Ph. D., Reading, Pa.

If three chords are drawn at random in a circle, what is the chance the center of the circle is enclosed by the three chords, and what is the mean area of this enclosing triangle?

203. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Find the average length of a hole at random through a given cone.

MISCELLANEOUS.

180. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Show that $\sum_{n=1}^{n=\infty}(-1)\frac{1}{(2n-1)(2^{2n}-1)}=\Sigma\tan^{-1}\frac{3.2^n}{2^{2n+1}-1}$, the smallest value of each inverse function being taken.

NOTES AND NEWS.

Dr. Herbert G. Keppel, of Northwestern University, has been elected head of the department of mathematics of the University of Florida.

The next annual meeting of the American Mathematical Society will be held at Baltimore, Md., during the holidays in connection with the annual conference of the American Society for the Advancement of Science.

At the meeting of the Central Association of Scienc and Mathematics Teachers, held in Chicago, November 27 and 28, 1908, a committee was appointed to select material for concrete applications in the teaching of geometry. This is a move in the right direction, for without doubt geometry teaching has become too much of an exercise in abstract logic.

A movement is under way whereby the recently organized Federation of Associations of Science and Mathematics Teachers throughout the

country may undertake definite work in the matter of educational improvement in these lines. One important step contemplated is the appointment of a representative committee to prepare a national syllabus on geometry.

In Science, for October 23rd, appears an article by Professor G. A. Miller, of the University of Illinois, on the "Method of Appointment of University Professors in Foreign Countries." This topic will also be discussed by Professor Wilczynski, of the University of Illinois, at the meet- of the Chicago Section of the American Mathematical Society to be held in Chicago, January 1 and 2, 1909.

At the recent meeting of the New York section of the Association of Teachhers of Mathematics in the Middle States and Maryland the general topic for discussion was:

"Modern Tendencies in the Teaching of Algebra, or An Evening with the Writers of Some of the Modern Text-books in Algebra."

The speakers were:

Fletcher Durell, Ph. D., Mathematical Master in the Lawrenceville School, joint author, with Edward R. Robbins, A. B., of the Durell and Robbins Series; Isaac J. Schwatt, Ph. D., Assistant Professor of Mathematics in the University of Pennsylvania, joint author with Dr. G. E. Fisher, of the Fisher and Schwatt Series; N. J. Lennes, Ph. D., Instructor in Mathematics, Brown University, joint author with Dr. H. E. Slaught, of the Slaught and Lennes Series.

The discussions were led by the following:

Mr. Wm. E. Breckenridge, Stuyvesant High School, New York City; Mr. Merle L. Bishop, Boys' High School, Brooklyn; Mr. Oscar W. Anthony, Dewitt Clinton High School, New York City.

This issue of the MONTHLY was mailed February 1.

Dr. T. W. Wright, author of a good text-book on Mechanics, Professor of Mathematics in Union College, and a subscriber of the MONTHLY from the first, died last September.

Mr. A. H. Bell, of Litchfield, Illinois, a valued contributor and subscriber of the MONTHLY from its beginning, is still doing civil engineering work though he is past 78 years of age.